DEPTH-WIDTH TRADE-OFFS FOR NEURAL NETWORKS VIA TOPOLOGICAL ENTROPY

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ABSTRACT. One of the central problems in the study of deep learning theory is to understand how the structure properties, such as depth, width and the number of nodes, affect the expressivity of deep neural networks. In this work, we show a new connection between the expressivity of deep neural networks and topological entropy from dynamical system, which can be used to characterize depth-width trade-offs of neural networks. We provide an upper bound on the topological entropy of neural networks with continuous semi-algebraic units by the structure parameters. Specifically, the topological entropy of ReLU network with *l* layers and *m* nodes per layer is upper bounded by $O(l\log m)$. Besides, if the neural network is a good approximation of some function *f*, then the size of the neural network has an exponential lower bound with respect to the topological entropy of *f*. Moreover, we discuss the relationship between topological entropy, the number of oscillations, periods and Lipschitz constant.

1. INTRODUCTION

Deep neural network has been a hot topic in machine learning, which has lots of applications ranging from pattern recognition to computer vision. Understanding the representation power of neural network is one of the key problems in deep learning theory. Universal approximation theorem tells us that any continuous function can be approximated by a depth-2 neural network with some activation function on a bounded domain [Cyb89,HMW89,Fun89,Bar94]. However, the size of the neural network in this approximation can be exponential which is impractical in real life. Hence, we are interested in the neural networks with bounded size.

One natural question is to investigate the trade-offs between depth and width. The benefits of depths on the representational power of neural networks has attracted lots of attention, and there are many results based on the depth separation

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argument [ES16, Tel15, Tel16, Sch00, MPCB14, MSS19, PLRDG16, RPJKGSD17, ABMM16, LS16, KTB19]. Depth separation argument has also been considered in other computational models, such as boolean circuits [Has86, Hås87, PGM94, RST15] and sum-product networks [DB11, MM14]. To get a depth separation argument for neural networks, several measures to quantify the complexity of the functions have been introduced, such as the number of linear regions [MPCB14], Fourier spectrum [ES16], global curvature [PLRDG16], trajectory length [RPJKGSD17], fractals [MSS19] and so on.

Recently, Telgarsky used the number of oscillations as a measure of the complexity of function to prove that there exist neural networks with $\theta(k^3)$ layers, $\theta(1)$ nodes per layer which can not be approximated by networks with O(k) layers and $O(2^k)$ nodes [Tel16]. Moreover, Chatziafratis et al provided a connection between the representation power of neural networks and the periods of the function by the well-known Sharkovsky's Theorem [CNPW19]. Furthermore, by revealing a tighter connection between periods, Lipschitz constant and the number of oscillations, Chatziafratis et al gave an improved depth-width trade-offs [CNP20].

In this work, we show the connection between the representation power of neural networks and topological entropy, a well-known concept in dynamic system to quantify the complexity of the system. First, we provide an upper bound on the topological entropy of neural networks with semi-algebraic units by the structure parameters like depth and width. For example, for the ReLU network with l layers and m nodes per layer, the topological entropy is upper bounded by $O(l\log m)$. Besides, if the neural network is a good approximation of some function f, then the size has an exponential lower bound with respect to the topological entropy of f. Furthermore, we discuss the connection between topological entropy, number of oscillations, periods and Lipschitz constant.

2. Preliminaries

2.1. **Background about dynamic system.** In this subsection, we will introduce some basic facts about one-dimensional dynamic system. First, let us introduce the definition of topological entropy. Topological entropy of a dynamic system quantifies the complexity of the system, such as the number of different orbits and the sensitivity of evolution on the initial states. There are several equivalent definitions of topological entropy. Here we take the one introduced by Adler, Konheim, and McAndrew [AKM65].

Let X be a compact Hausdorff space, f be a continuous map from X to X. Given a set \mathscr{A} of subsets of X, if their union is X, then \mathscr{A} is called a cover of X. If each element in \mathscr{A} is an open set, then \mathscr{A} is called an open cover of X. Given open covers $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n$ of X, we denote $\bigvee_{i=1}^n \mathscr{A}_i$ as follows,

$$\bigvee_{i=1}^{n} \mathscr{A}_{i} := \{A_{1} \cap A_{2} \dots \cap A_{n} : A_{i} \in \mathscr{A}_{i}, \forall i, \text{ and } A_{1} \cap A_{2} \dots \cap A_{n} \neq \emptyset \}.$$

Given an open cover \mathscr{A} , we can define the open cover $f^{-i}(\mathscr{A})$ and \mathscr{A}_f^n as follows

$$f^{-i}(\mathscr{A}) := \{f^{-i}(A) : A \in \mathscr{A}\},\$$
$$\mathscr{A}_{f}^{n} = \bigvee_{i=0}^{n-1} f^{-i}(\mathscr{A}).$$

Let us denote $\mathscr{N}(\mathscr{A})$ to be the minimal cardinality of the subcover from \mathscr{A} . Mathematically, $\mathscr{N}(\mathscr{A})$ can be defined as follows

$$\mathcal{N}(\mathscr{A}) = \min \{ Card(\mathscr{B}) : \mathscr{B} \subset \mathscr{A} \text{ and } \mathscr{B} \text{ is a cover of } X \},\$$

where $Card(\mathscr{B})$ denotes the cardinality of \mathscr{B} .

Now, we are ready to define topological entropy.

Definition 1. [AKM65] Given a compact Hausdorff topological space X, and a continuous map $f: X \to X$, for an open cover \mathscr{A} , the topological entropy of f on the cover \mathscr{A} is defined as

$$h_{top}(f,\mathscr{A}) = \lim_{n \to \infty} \frac{1}{n} \log_2 \mathscr{N}(\mathscr{A}_f^n).$$

The topological entropy of f is defined as

$$h_{top}(f) = \sup_{\mathscr{A}: \text{ open cover of } X} h_{top}(f, \mathscr{A}).$$
(1)

The topological entropy takes value from $[0, +\infty]$. (See Fig 1 for the examples of functions with finite and infinite topological entropy.) Topological entropy has some nice properties, which we have listed in the Appendix A. In this work, we consider the case where X is a closed interval [a,b] and f is a continuous function from [a,b] to [a,b]. For such interval map, topological entropy has several nice characterization. In this work, we will use the following one. We list other characterizations in Appendix A.

Definition 2. A continuous function $f : [a,b] \to [a,b]$ is piece-wise monotone, if there exists a finite partition of [a,b] such that f is monotone on each piece. Let us denote c(f) to be minimal number of monotonicity of f.

Lemma 3. [*Mis80b*, You81] If the continuous function $f : [a,b] \rightarrow [a,b]$ is piecewise monotone, then

$$h_{top}(f) = \lim_{n \to \infty} \frac{1}{k} \log c(f^k) = \inf_k \frac{1}{k} \log c(f^k),$$

where c(f) is the number of intervals of monotonicity of f.

Now let us introduce the definition of periods in the dynamical system.

Definition 4. A continuous function $f : [a,b] \rightarrow [a,b]$ has a point of period *n* if there exists $x_0 \in [a,b]$ such that

$$f^{n}(x_{0}) = x_{0},$$

$$f^{i}(x_{0}) \neq x_{0}, \forall 1 \le i \le n-1$$

The set $\{x_0, f(x_0), \dots, f^{n-1}(x_0)\}$ is called a *n*-cycle of *f*.



FIGURE 1. Examples of functions with finite and infinite topological entropy. (a) $g: [0,1] \rightarrow [0,1]$ with $h_{top}(g) = 3$; (b) $f: [0,1] \rightarrow [0,1]$ with $h_{top}(f) = +\infty$, where f is conjugate to g^n on the interval $[2^{-(n-1)}, 2^{-n}]$ for each integer $n \ge 0$ and f(0) = 0. (See the definition of conjugacy in Appendix A.)

There is a well-known theorem called Sharkovsky's Theorem, which describes the structure of the periods of cycles of the interval map.

Definition 5 (Sharkovsky's ordering). Let us define Sharkovsky ordering as follows

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd$$
$$\rhd 3 \cdot 2 \rhd 5 \cdot 2 \rhd 7 \cdot 2 \rhd \cdots \rhd$$
$$\Rightarrow 3 \cdot 2^{2} \rhd 5 \cdot 2^{2} \rhd 7 \cdot 2^{2} \rhd \cdots \rhd$$
$$\vdots$$
$$\Rightarrow 3 \cdot 2^{n} \rhd 5 \cdot 2^{n} \rhd 7 \cdot 2^{n} \rhd \cdots \rhd$$
$$\vdots$$
$$\vdots$$
$$\rhd \cdots \rhd 2^{3} \rhd 2^{2} \rhd 2 \rhd 1$$

Let us define Per(f) to be the set of periods of cycles of a map $f : [a,b] \to [a,b]$ and denote $\mathbb{N}_{sh} = \mathbb{N} \cup \{2^{\infty}\}$. Sharkovsky's Theorem tells us that Sharkovsky's ordering can be used to characterize the periods of a continuous function as follows.

Theorem 6. [Sha64, Sha65] Given a continuous function $f : [a,b] \to [a,b]$, there exists $s \in \mathbb{N}_{sh}$ such that $Per(f) = \{k \in \mathbb{N} : s \triangleright k\}$. Conversely, for any $s \in \mathbb{N}_{sh}$, there exists a continuous function $f : [a,b] \to [a,b]$ such that $Per(f) = \{k \in \mathbb{N} : s \triangleright k\}$.

Next, let us give the definition of crossings (or oscillations), where the relationship between the number of crossings and periods has been considered in [CNPW19, CNP20].

Definition 7. Given a continuous function $f : [a,b] \rightarrow [a,b]$, for any $[x,y] \subset [a,b]$, f crosses [x,y] if there exists $c,d \in [a,b]$ such that f(c) = x, f(d) = y. We use $C_{x,y}(f)$ to denote the number that f crosses [x,y], which means there exists $c_1,d_1 < c_2,d_2 < ... < c_t,d_t$ with $t = C_{x,y}(f)$ such that $f(c_i) = x, f(d_i) = y$ for any $1 \le i \le C_{x,y}(f)$.

Finally, let us introduce the concept called *f*-covering [ALM00].

Definition 8 (*f*-covering). Given a continuous function $f : [a,b] \rightarrow [a,b]$ and two intervals $I_1, I_2 \subset [a,b]$, we say that I_1 *f*-covers I_2 if there exists a subinterval *J* of I_1 such that $f(J) = I_2$. Besides, we say that I_1 *f*-covers I_2 *t* times if there exists *t* subintervals $J_1, ..., J_t$ of I_1 with pairwise disjoint interior such that $f(J_i) = I_2$ for i = 1, ..., t.

Based on the definitions of crossing and *f*-covering, it is easy to see that $C_{x,y}(f) = t$ iff the maximal times that [a,b] *f*-covers [x,y] is equal to *t*.

2.2. Neural networks with semi-algebraic units. A neural network is a function defined by a connected directed graph with some activation function $\sigma : \mathbb{R} \to \mathbb{R}$ and a set of parameters: a weight for each edge and a bias for each node of the graph. Usually the activation function $\sigma : \mathbb{R} \to \mathbb{R}$ is a nonlinear function. The root nodes do the computation on the input vector, while the internal nodes do the computation on the output from other nodes. The activation function for nodes may be different, and there are two common choices: (1) ReLU gate: $\vec{x} \to \sigma_R(\langle \vec{a}, \vec{x} \rangle + b)$, where $\sigma_R(x) = \max\{0, x\}$; (2) maximaization gate $Max: \vec{x} \to \max_{i=1}^{n} x_i$.

Here we consider an important class of activation functions, called semi-algebraic units (or semi-algebraic gates) [Tel16]. The definition of a semi-algebraic gate is given as follows

Definition 9. A function $\sigma : \mathbb{R}^n \to \mathbb{R}$ is called (t, d_1, d_2) semi-algebraic, if there exists *t* polynomials $\{p_i\}_{i=1}^t$ of degree $\leq d_1$ and *s* tripes $(L_j, U_j, q_j)_{j=1}^s$ where L_i and U_i are subsets of $\{1, 2, ..., t\}$, and each q_j is a polynomial of degree $\leq d_2$ such that

$$f(\vec{x}) = \sum_{j=1}^{3} q_j(\vec{x}) \left(\prod_{i \in L_j} \mathbb{I}(p_i(\vec{x}) < 0) \right) \left(\prod_{i \in U_j} \mathbb{I}(p_i(\vec{x}) < 0) \right),$$
(2)

where $\mathbb{I}(\cdot)$ is the indicator function.

Here, we are interested in the continuous semi-algebraic unit, that is the function $\sigma : \mathbb{R}^n \to \mathbb{R}$ is continuous and semi-algebraic. For example, the standard ReLU gate $\vec{x} \to \sigma_R(\langle \vec{a}, \vec{x} \rangle + b)$ is a continuous and (1, 1, 1) semi-algebraic unit [Tel16]. The maximization gate $Max : \mathbb{R}^n \to \mathbb{R}$ defined as $Max(\vec{x}) = \max_{i=1}^n x_i$ is a continuous and (n(n-1), 1, 1) semi-algebraic unit [Tel16].

Definition 10. A function $\sigma : \mathbb{R} \to \mathbb{R}$ is called (t, d)-poly, if there exists a partition of \mathbb{R} into $\leq t$ intervals such that σ is a polynomial of degree $\leq d$ on each interval.

Denote $\mathcal{N}_n(l, m, t, d_1, d_2)$ to be the set of neural networks with $\leq l$ layers, $\leq m$ nodes per layer, the activation function being continuous and (t, d_1, d_2) semialgebraic and the input dimension being *n*. As the function *f* we would like to represent is a continuous function $f : [a,b] \to [a,b]$, we consider the neural networks with input dimension being 1, i.e., $\mathcal{N}_1(l,m,t,d_1,d_2)$.

3. INFORMAL STATEMENT OF OUR MAIN RESULTS

Our first result shows the connection between topological entropy and the depth, width of deep neural networks, and provides an upper bound of the topological entropy of neural networks with continuous semi-algebraic units by the structure parameters.

Theorem 11 (Informal version of Theorem 14). For any neural networks g with l layers, m nodes per layer and (t, d_1, d_2) semi-algebraic units as activation function, then

$$h_{top}(\tau \circ g) \le l(1 + \log_2 m + \log_2 t + \log_2 d_1) + l^2 \log_2 d_2,$$
(3)

where $\tau : \mathbb{R} \to \mathbb{R}$ is defined as follows (See Figure 2.)

$$\tau(x) = \begin{cases} a, x \le 1, \\ x, a \le x \le b \\ b, x > b. \end{cases}$$
(4)



FIGURE 2. The figure for the function $\tau(x)$.

Our second result shows the connection between the topological entropy of a given function f and the depth-width trade-offs required to have a good approximation of f.

Theorem 12 (Informal statement of Theorem 16). *Given a continuous function* $f : [a,b] \rightarrow [a,b]$ with positive and finite topological entropy, if g is a good approximation of f with respect to $\|\cdot\|_{L^{\infty}}$, where g is a neural network with l layers, m nodes per layer and (t,d_1,d_2) semi-algebraic units as activation function, then we have

$$m \ge \frac{exp(\Omega(\frac{1}{l}h_{top}(f)))}{2td_1d_2^l}.$$
(5)

Hence, if the neural network g is a good approximation of f^k *with respect to* $\|\cdot\|_{L^{\infty}}$ *, then we have*

$$m \ge \frac{exp(\Omega(\frac{k}{l}h_{top}(f)))}{2td_1d_2^l}.$$
(6)

Our third result discusses the connection between the topological entropy, periods, the number of oscillations and Lipschitz constant.

4. CONNECTION BETWEEN TOPOLOGICAL ENTROPY AND THE SIZE OF NEURAL NETWORKS

First, let us consider the topological entropy of the neural networks with l layers, m nodes per layer and activation function being (t, d_1, d_2) semi-algebraic and continuous, i.e., the functions from $\mathcal{N}_1(l, m, t, d_1, d_2)$. Let us define $\tau : \mathbb{R} \to \mathbb{R}$ as follows

$$t_2(x) = \begin{cases} a, & x \leq 1, \\ x, & a \leq x \leq b, \\ b, & x > b. \end{cases}$$

We can rewrite $\tau(x)$ as follows

$$\tau(x) = a + (x - a)\mathbb{I}(x > a) + (b - x)\mathbb{I}(x > b).$$
(7)

Hence τ is continuous and (2, 1, 1) semi-algebraic. Therefore, for any $g \in \mathcal{N}_1(l, m, t, d_1, d_2)$, the function $\tau \circ g$ is a continuous function from [a, b] to [a, b]. Thus, we can compute the topological entropy of $\tau \circ g$.

To get an upper bound on the topological entropy of neural networks, we first need the following lemma, which gives an upper bound on the number of intervals of monotonicity of f.

Lemma 13. If the function
$$f : [a,b] \to [a,b]$$
 is continuous and (t,d) -poly, we have $c(f) \le td$. (8)

Proof. Since $f : [a,b] \to [a,b]$ is (t,d)-poly, then there exists a partition of the interval [a,b] into subintervals $\{J_i\}_{i=1}^t$ such that f is a polynomial of degree $\leq d$ on each subinterval J_i . It is directly for any polynomial degree $\leq d$, we can divide \mathbb{R} into $\leq d$ intervals such that this polynomial is monotone in each piece. Hence, we can divide each subinterval J_i into at most d pieces, such that f is monotone on each piece. Thus

$$c(f) \leq td.$$

Now, we are ready to prove our first result, which gives an upper bound on the topological entropy of the neural networks by the structure parameters of the neural networks.

Theorem 14. For any $g \in \mathcal{N}_1(l, m, t, d_1, d_2)$, the topological entropy for the function $\tau \circ g : [a,b] \to [a,b]$ is upper bounded by the structure parameters as follows

$$h_{top}(\tau \circ g) \le l(1 + \log_2 m + \log_2 t + \log_2 d_1) + 2l^2 \log d_2.$$
(9)

Proof. It has been proved that if the function $f : \mathbb{R}^n \to \mathbb{R}$ is (t, d_1, d_2) semi-algebraic, $g_1, ..., g_n : \mathbb{R} \to \mathbb{R}$ is (s, d_3) -poly, then $\mu(x) := f(g_1(x), ..., g_n(x))$ is $(stn(1+d_1d_3), d_2d_3)$ -poly [Tel16]. Thus, by analyzing the neural network layer by layer, for any $g \in \mathcal{N}_1(l, m, t, d_1, d_2), \tau \circ g$ is (α_l, β_l) -poly, where

$$\begin{aligned} \alpha_l &\leq 2(2mtd_1)^l d_2^{\frac{1}{2}l^2+l}, \\ \beta_l &\leq d_2^l. \end{aligned}$$

Therefore, by Lemma 13, we have

$$c(\tau \circ g) \leq 2(2mtd_1)^l d_2^{2l^2}.$$

By Lemma 3, we have

$$\lim_{k} \frac{1}{k} \log_2 c(f^k) = \inf_{k} \frac{1}{k} \log_2 c(f^k) = h_{top}(f),$$

which implies that

$$c(f) \ge 2^{h_{top}(f)}$$

Therefore, we have

$$h_{top}(\tau \circ g) \le l(1 + \log_2 m + \log_2 t + \log_2 d_1) + 2l^2 \log_2 d_2.$$

Next, to get the relationship between topological entropy of the function f and that of the neural networks, we need to consider the continuity of the topological entropy.

Lemma 15. [*Mis79*] For any continuous function $f : [a,b] \to [a,b]$, it holds that $\liminf_{g \to f} h_{top}(g) \ge h_{top}(f), \tag{10}$

where $g : [a,b] \to [a,b]$ is continuous and $g \to f$ by L^{∞} norm.

Based on the lower semi-continuity of topological entropy, if the given function has finite topological entropy, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any continuous function $g: [a,b] \to [a,b]$ with $||f-g||_{L^{\infty}} < \delta$, we have

$$h_{top}(g) \ge h_{top}(f) - \varepsilon.$$

If $0 < h_{top}(f) < +\infty$, let us take $\varepsilon = \frac{1}{2}h_{top}(f)$, there exists $\delta(f) > 0$ such that for any continuous function g with $||f - g||_{L^{\infty}} < \delta(f)$, we have

$$h_{top}(g) \ge \frac{1}{2}h_{top}(f).$$

Theorem 16. Given a continuous function $f : [a,b] \to [a,b]$ with positive and finite topological entropy, then there exists $\delta(f) > 0$ such that for any $g \in \mathcal{N}_1(l,m,t,d_1,d_2)$ with $||f-g||_{L^{\infty}} \leq \delta(f)$, we have

$$m \ge \frac{2^{\frac{1}{2l}h(f)}}{2td_1 d_2^{2l}}.$$
(11)

Proof. First, based on Lemma 15, there exists $\delta(f) > 0$ such that for any continuous function $g : [a,b] \to [a,b]$, we have

$$h_{top}(\tau \circ g) \geq \frac{1}{2}h_{top}(f).$$

Besides, it is easy to see that τ is a Lipschitz function and $|\tau(x) - \tau(y)| \le |x - y|$. Hence, for any $g \in \mathcal{N}_1(l, m, t, d_1, d_2)$ with $||f - g||_{L^{\infty}} \le \delta(f)$, we have

$$\|\tau\circ g-f\|_{L^{\infty}}\leq \|g-f\|_{L^{\infty}}\leq \delta(f).$$

Then the topological entropy of $\tau \circ g : [a,b] \to [a,b]$ has the following lower bound,

$$h_{top}(\tau \circ g) \ge \frac{1}{2} h_{top}(f)$$

However, due to Theorem 14, for any $g \in \mathcal{N}_1(l, m, t, d_1, d_2)$, we have

 $h(\tau \circ g) \le 1 + l + l \log_2 m.$

Therefore, we have

$$\frac{1}{2}h_{top}(f) \le l(1 + \log_2 m + \log_2 t + \log_2 d_1) + 2l^2 \log_2 d_2$$

That is

$$m \ge \frac{2^{\frac{1}{2l}h_{lop}(f)}}{2td_1d_2^{2l}}.$$

Theorem 16 tells us that if the neural network $g \in \mathcal{N}_1(l, m, t, d_1, d_2)$ is a good approximation (i.e., $||f - g||_{L^{\infty}} \leq \delta(f)$), then the depth *m* has an exponential lower bound with respect to the topological entropy.

Besides, if we iterate the function for k times, i.e, f^k and the neural network $g \in \mathcal{N}_1(l, m, t, d_1, d_2)$ is a good approximation of f^k , we have the following corollary.

Corollary 17. Given a continuous function $f : [0,1] \rightarrow [0,1]$ with positive and finite topological entropy, then there exists $\delta(f^k) > 0$ such that for any $g \in \mathcal{N}_1(l,m,t,d_1,d_2)$ with $\|f^k - g\|_{L^{\infty}} \leq \delta(f^k)$, we have

$$m \ge \frac{2\frac{k_{2l}}{2l}h_{top}(f)}{2td_1d_2^{2l}}.$$
(12)

Proof. This corollary comes directly from Theorem 16 and $h_{top}(f^k) = kh_{top}(f)$ for any integer $k \ge 0$. (See Lemma 24 in Appendix A.)

For example, if we take the activation function to be ReLU unit which is continuous and (1,1,1) semi-algebraic, then the following statements come directly from Theorem 14 and 16.

Proposition 18. For any ReLU network g with at most l layers and at most m nodes per layer, then

$$h_{top}(\tau \circ g) \le l(1 + \log_2 m). \tag{13}$$

Proposition 19. Given a continuous function $f : [a,b] \to [a,b]$ with finite topological entropy, then there exists $\delta(f) > 0$ such that for any ReLU network g with at most l layers and at most m nodes per layer which satisfies $||f - g||_{L^{\infty}} \leq \delta(f)$, we have

$$m > 2^{\frac{1}{2l}h_{lop}(f)-1}.$$
(14)

Moreover, if g is a good approximation of f^k with respect to L^{∞} norm, i.e., $\|f^k - g\|_{L^{\infty}} \leq \delta(f^k)$, then we have

$$m \ge 2^{\frac{k}{2l}h_{top}(f)-1}.$$
 (15)

If the function f we would like to present has infinity topological entropy, i.e., $h_{top}(f) = +\infty$, then due to the lower semi-continuity of topological entropy, for any N > 0, there exists $\delta_N(f) > 0$ such that for any continuous function $g : [a,b] \rightarrow [a,b]$ with $||g - f||_{L^{\infty}} < \delta_N(f)$,

$$h_{top}(g) \ge N \tag{16}$$

Proposition 20. Given a continuous function $f : [a,b] \to [a,b]$ with $h_{top}(f) = +\infty$, then any N > 0 sufficiently large, there exists $\delta_N(f) > 0$ such that for any $g \in \mathcal{N}_1(l,m,t,d_1,d_2)$ with $||f-g||_{L^{\infty}} < \delta_N(f)$, we have

$$m \ge \frac{2^{N/l}}{2td_1d_2^l}.$$
 (17)

Proof. The proof is the same as Theorem 16.

4.1. **Examples.** First, let us consider the tent map $t_{\alpha} : [0,1] \rightarrow [0,1]$, where $t_{\alpha}(x)$ is defined as follows

$$t_{\alpha}(x) = \begin{cases} \alpha x, 0 \le x \le 1/2, \\ \alpha(1-x), 1/2 < x \le 1, \end{cases}$$

where $0 \le \alpha \le 2$. (See Figure 3)



FIGURE 3. Tent map t_{α} and t_{α}^4 with different parameters α .

The topological entropy of t_{α} can be easily computed by Lemma 30, and we have

$$h_{top}(t_{\alpha}) = \begin{cases} 0, 0 \le \alpha \le 1, \\ \log_2 \alpha, 1 < \alpha \le 2. \end{cases}$$





FIGURE 4. The topological entropy of the tent map t_{α} for $0 < \alpha \leq 2$.

Hence, based on Theorem 16, if we would like to have a good approximation of t_{α}^{k} for $\alpha > 1$, then the width required to represent t_{α}^{k} with continuous and (t, d_{1}, d_{2}) semi-algebraic units is

$$m \ge C(t, d_1, d_2) \alpha^{k/l},$$

where $C(t, d_1, d_2)$ is a constant which only depends on t, d_1, d_2 .

Next, let us consider the logistic map $f_{\beta} : [0,1] \rightarrow [0,1]$ as follows

$$f(x) = \beta x(1-x),$$

where the parameter β is taken from [0,4] (See Figure 5). Logistic map has been used to get lower bounds on the size of sigmoidal neural networks [Sch00].



FIGURE 5. Logistic map f_{β} and f_{β}^4 with different parameters β .

It is easy to see that $h_{top}(f_{\beta}) = 1$ when $\beta = 4$, and $h_{top}(f_{\beta}) = 0$ when $\beta = 2$. Hence, based on Theorem 16, if we would like to have a good approximation of f_4^k , then the width required to represent f_4^k with continuous and (t, d_1, d_2) semi-algebraic function is

$$m \ge C(t, d_1, d_2) 2^{k/l},$$

where $C(t, d_1, d_2)$ is a constant which only depends on t, d_1, d_2 .

5. RELATIONSHIP BETWEEN TOPOLOGICAL ENTROPY AND PERIODS, NUMBER OF CROSSINGS AND LIPSCHITZ CONSTANT

In this section, we will discuss the connection between topological entropy and periods, number of crossings and Lipschitz constant.

5.1. Relationship between topological entropy and periods, the number of crossings. In fact, the relationship between topological entropy and periods has been discussed in [ALM00], which has the following statement.

Lemma 21 ([ALM00]). *Given a continuous map* $f : [a,b] \rightarrow [a,b]$ *, it has positive topological entropy iff it has a cycle of period which is not a power of 2.*

In this subsection, we will show the connection between topological entropy and the number of crossings for piece-wise monotone function $f : [a,b] \rightarrow [a,b]$. Let us define C(f) as follows

$$C(f) := \sup_{x < y} C_{x,y}(f), \tag{18}$$

which is the maximal number of crossings over any interval $[x,y] \subset [a,b]$. We find the relationship between the maximal number of crossings C(f) and topological entropy $h_{top}(f)$ in the asymptotic case.

Proposition 22. *Given a continuous function* $f : [a,b] \rightarrow [a,b]$ *which is piece-wise monotone, then*

$$\lim_{k \to \infty} \sup_{k} \frac{1}{k} \log_2 C(f^k) = h_{top}(f).$$
⁽¹⁹⁾

Proof. First, since f is piece-wise monotone, then there exists a finite partition of [a,b] into subintervals such that f is monotone on each subinterval. For any subinterval where f is monotone, there is at most one crossing over [x,y]. Thus for any $x, y \in [a,b]$, we have

$$C_{xy}(f) \le c(f),$$

i.e., $C(f) \leq c(f)$. Therefore,

$$\lim_{k} \sup_{k} \frac{1}{k} \log_2 C(f^k) \le \lim_{k} \frac{1}{k} \log_2 C(f^k) = h_{top}(f).$$

Besides, if $h_{top}(f) = 0$, then we have already got the result as

$$\lim_{k \to \infty} \sup_{k} \frac{1}{k} \log_2 C(f^k) \ge 0$$

Hence, we only need to consider the case where $h_{top}(f) > 0$. Let us introduce the concept called *s*-horeses [Mis79, Mis80a], which is an interval $J \subset [a,b]$ and a partition \mathscr{D} of *J* into *s* subintervals such that the closure of each element of \mathscr{D} *f*-covers *J*. It has been proved in [Mis79, Mis80a] that there exist sequences $\{k_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of positive integers such that $\lim_{n\to\infty} k_n = \infty$ and for each *n*, there exists s_n -horseshoes (J_n, D_n) for f^{k_n} such that

$$\lim_{n\to\infty}\frac{1}{k_n}\log_2 s_n = h_{top}(f).$$

Based on the definition of s_n -horseshoe, for the map f^{k_n} , the closure of each subinterval in D_n f^{k_n} -covers J_n . Thus, based on the definition of crossings, we have

$$C(f^{k_n}) \geq C_{J_n}(f^{k_n}) \geq s_n.$$

Therefore

$$\lim_{k} \sup_{k} \frac{1}{k} \log_2 C(f^k) \ge \lim_{n \to \infty} \frac{1}{k_n} \log_2 s_n = h_{top}(f).$$

5.2. Relationship between topological entropy and Lipschitz constant. Let us consider the connection between Lipschitz constant and topological entropy. Let us denote the Lipschitz constant of f by L(f), that is

$$L(f) = \inf \{ L \ge 0 : |f(x) - f(y)| \le L|x - y|, \forall x, y \in [a, b] \}.$$
 (20)

The connection between periods, the number of crossings and Lipschitz constant has been discussed in [CNP20]. It has been proved that if the Lipschitz constant matches the number of crossings, i.e., $C_{xy}(f^k) = L(f^k)$, then a L^1 -separation between f^k and ReLU neural networks can be obtained [CNP20]. Here we discuss the relationship between Lipschitz constant and topological entropy.

Proposition 23. Given a continuous function $f : [a,b] \rightarrow [a,b]$ which piece-wise monotone, then

$$\lim_{k \to \infty} \frac{1}{k} \log_2 L(f^k) = \inf_k \frac{1}{k} \log_2 L(f^k), \tag{21}$$

and

$$\lim_{k \to \infty} \max\left\{0, \frac{1}{k} \log_2 L(f^k)\right\} \ge h_{top}(f).$$
(22)

Proof. Based on the definition of Lipschitz constant, it is easy to see that

$$\begin{aligned} |f^{n+k}(x) - f^{n+k}(y)| &= |f^n(f^k(x)) - f^n(f^k(y))| \\ &\leq L(f^n)|f^k(x) - f^k(y)| \\ &\leq L(f^n)L(f^k)|x - y|, \end{aligned}$$

for any integers n, k and any $x, y \in [a, b]$. Thus,

$$L(f^{n+k}) \le L(f^n)L(f^k).$$
(23)

i.e., $\log_2 L(f^{n+k}) \leq \log_2 L(f^n) + \log_2 L(f^k)$. Hence $\{\log_2 L(f^k)\}_k$ is a subadditive sequence. Therefore, according to Lemma 31 in Appendix A, the limit

$$\lim_{k\to\infty}\frac{1}{k}\log_2 L(f^k)$$

exists and

$$\lim_{k \to \infty} \frac{1}{k} \log_2 L(f^k) = \inf_k \frac{1}{k} \log_2 L(f^k).$$

Let us another characterization of topological entropy of the function, which is piece-wise monotone, by variation [ALM00] as follows

$$\lim_{k \to \infty} \max\left\{0, \frac{1}{k} \log_2 Var(f^k)\right\} = h_{top}(f),$$

where variation Var(f) is defined to be the supremum of

$$\sum_{i=1}^{t} |f(x_{i+1}) - f(x_i))|,$$

over all finite sequences $x_1 < x_2 < ... < x_t$ in [a,b]. (See Lemma 29 in Appendix A.) Due to the definition of Var(f), it is easy to see

$$Var(f^k) \le L(f^k)|b-a|.$$

Therefore,

$$\lim_{k \to \infty} \frac{1}{k} \log_2 L(f^k) \ge \lim_{k \to \infty} \frac{1}{k} \log_2 Var(f^k),$$

which implies that

$$\lim_{k\to\infty} \max\left\{0, \frac{1}{k}\log_2 L(f^k)\right\} \ge h_{top}(f).$$

Based on Proposition 23, if $L(f^k) \ge 1$, then $L(f^k)$ has an exponential lower bound with respect to the topological entropy of f, i.e., $L(f^k) \ge 2^{kh_{top}(f)}$.

6. CONCLUSION

In this paper, we have investigated the relationship between topological entropy and expressivity of deep neural networks. We provide a depth-width trade-offs based on the topological entropy from the theory of dynamic system. For example, the topological entropy of the ReLU network with *l* layers and *m* nodes per layer is upper bounded by $O(l\log m)$. Besides, we show that the size of the neural network required to represent a given function has an exponential lower bound with respect to the topological entropy of the function, where the exponential lower bound holds for L^{∞} -error approximation. For example, if we would like to represent the function *f* by ReLU network with *l* layers and *m* nodes per layer, then the width *m* has a lower bound $\exp(\Omega(h_{top}(f)/l))$. Moreover, we discuss the relationship between topological entropy, periods, Lipschitz constant and the number of crossings, especially the relationship in the asymptotic case.

Note that one key step to get exponential lower bound on the size of neural networks for L^{∞} -error approximation is the lower semi-continuity of topological entropy with respect to L^{∞} norm. If the lower semi-continuity of topological entropy with respect to L^{p} norm (e.g., L^{1} norm) holds, it will lead to exponential lower bound (with respect to topological entropy) for L^{p} -error approximation. Further studies on the (semi-)continuity of topological entropy are desired. Besides, it would be quite interesting to study the relationship between topological entropy and VC dimension. We leave it for further study.

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APPENDIX A. PROPERTIES OF TOPOLOGICAL ENTROPY

Here, we list some useful facts about topological entropy. More information can be found in [ALM00].

Lemma 24. [ALM00] Given a compact Hausdorff space X and a continuous function $f: X \to X$, topological entropy of f and f^k has the following relationship

$$h_{top}(f^k) = kh_{top}(f), \tag{24}$$

for any integer $k \ge 0$.

Proposition 25. [*ALM00*] *Given compact Hausdorff spaces* $X, Y, f : X \to X, g : Y \to Y, \phi : X \to Y$ are continuous maps such that the following diagram

commutes, i.e., $\varphi \circ f = g \circ \varphi$ *, we have the following properties*

(a) if φ is injective, then $h_{top}(f) \leq h_{top}(g)$,

(b) if φ is surjective, then $h_{top}(f) \ge h_{top}(g)$,

(c) if φ is bijective, then $h_{top}(f) = h_{top}(g)$. And φ is called a conjugacy between f and g (or f and g are conjugate).

If X = [a, b], then we have the following characterization of topological entropy for a continuous function $f : [a, b] \rightarrow [a, b]$.

Definition 26 ([Mis79, Mis80a]). Given a continuous function $f : [a,b] \rightarrow [a,b]$, an s-horseshoe with $s \ge 2$ for f is (J, \mathcal{D}) , where $J \subset [a,b]$ is an interval and \mathcal{D} is a

partition J into s subintervals such that the closure of each element of \mathcal{D} f-covers J.

Lemma 27 ([Mis79, Mis80a]). Given a continuous function $f : [a,b] \to [a,b]$ with positive entropy, then there exist sequences $\{k_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of positive integers such that $\lim_{n\to\infty} k_n = \infty$, for each n the map f^{k_n} has an s_n -horseshoe and

$$\lim_{n \to \infty} \frac{1}{k_n} \log s_n = h_{top}(f).$$
(26)

Definition 28. [ALM00] Given a continuous function $f : [a,b] \rightarrow [a,b]$, the variation Var(f) is defined to be the supremum of

$$\sum_{i=1}^{t} |f(x_{i+1}) - f(x_i))|$$
(27)

over all finite sequences $x_1 < x_2 < \dots < x_t$ in [a, b].

Lemma 29. [*ALM00*] Given a continuous function $f : [a,b] \rightarrow [a,b]$ which piecewise monotone, then we have

$$\lim_{k \to \infty} \max\left\{0, \frac{1}{k} \log_2 Var(f^k)\right\} = h_{top}(f).$$
(28)

Lemma 30. [*Mis80b*] Given a continuous function $f : [a,b] \rightarrow [a,b]$, which is piece-wise monotone, if f is affine with the slope coefficient of absolute value s on each piece of monotonicity, then

$$h_{top}(f) = \max\{0, \log_2 s\}.$$
 (29)

Lemma 31. [ALM00] Given a subadditive sequence $\{a_k\}_{k=1}^{\infty}$ (i.e. $a_{n+k} \leq a_n + a_k$), we have

$$\lim_{k \to \infty} \frac{a_k}{k} \tag{30}$$

exists and is equal to $\inf_k \frac{a_k}{k}$.