JOURNAL OF FUNCTIONAL ANALYSIS **41**, 290–306 (1981)

Entropy Numbers, s-Numbers, and Eigenvalue Problems

BERND CARL

Sektion Mathematik, Universität Jena, Universitätshochhaus, DDR-69 Jena, German Democratic Republic

Communicated by the Editors

Received March 30, 1980

We establish inequalities between entropy numbers and approximation numbers for operators acting between Banach spaces. Furthermore we derive inequalities between eigenvalues and entropy numbers for operators acting on a Banach space. The results are compared with the classical inequalities of Bernstein and Jackson.

0. INTRODUCTION

Let $\mathscr{L}(E, F)$ denote the set of all (bounded linear) operators from the Banach space E into the Banach space F.

For every operator $S \in \mathscr{L}(E, F)$ the nth entropy number $e_n(S)$ is defined to be the infimum of all $\varepsilon \geqslant 0$ such that there are $y_1, \ldots, y_{2^{n-1}} \in F$ for which

$$S(U_E) \subseteqq \bigcup_{1}^{2^{n-1}} \{ y_i + \varepsilon U_F \}$$

holds.

Here U_E and U_F are the closed unit balls of E and F, respectively.

The theory of entropy numbers was introduced and studied by Pietsch in [15, (12)]. However, certain functions inverse to the ε-entropy appeared in the work of Mitjagin and Pełczyński [13] and Triebel [22].

Roughly speaking, the asymptotic behaviour of $e_n(S)$ characterizes the "degree of compactness" of S. In particular, S is compact if and only if $\lim e_n(S) = 0$.

The entropy numbers have the following nice properties of an additive and multiplicative s-number function [15, (12)]:
Monotonicity:

$$\|S\| = e_1(S) \geqslant e_2(S) \geqslant \cdots \geqslant 0 \qquad \text{for} \quad S \in \mathscr{L}(E, F);$$

additivity:

$$e_{n+m-1}(S + T) \leqslant e_n(S) + e_m(T) \qquad \text{for} \quad S, T \in \mathscr{L}(E, F);$$

0022-1236/81/060290–17\$02.00/0

multiplicativity:

$$e_{n+m-1}(ST) \leqslant e_n(S)\, e_m(T) \qquad \text{for} \quad T \in \mathscr{L}(E, F), S \in \mathscr{L}(F, G).$$

Put

$$\mathscr{L}_{p,q}^{(e)} := \{S \in \mathscr{L}: (e_n(S)) \in l_{p,q}\}$$

and

$$L_{p,q}^{(e)}(S) := \varepsilon_{p,q} \, \|(e_n(S))\|_{p,q} \qquad \text{for} \quad S \in \mathscr{L}_{p,q}^{(e)},$$

where $[l_{p,q}; \| \ \|_{p,q}]$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, stands for the quasinormed Lorentz sequence spaces (cf. [17]) and $\varepsilon_{p,q}$ is a norming constant (cf. [15, (14.3)]). Then $[\mathscr{L}_{p,q}^{(e)}; L_{p,q}^{(e)}]$ becomes an injective and surjective quasinormed operator ideal [15, (14.3.5)].

From the multiplicativity of the entropy numbers we get the useful product formula

$$\mathscr{L}_{p_2,q_2}^{(e)} \circ \mathscr{L}_{p_1,q_1}^{(e)} \subseteq \mathscr{L}_{p,q}^{(e)} \qquad \text{for} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

This definition of the product of quasinormed operator ideals comes from [15, (7.1)].

For our purpose we need the following s-numbers: If $S \in \mathscr{L}(E, F)$ and $n = 1, 2, \ldots$, then the nth approximation number, Gelfand number, and Kolmogorov number are defined by

$$a_n(S) := \inf\{\|S - L\|: \operatorname{rank}(L) < n\},$$

$$c_n(S) := \inf\{\|SJ_M^E\|: \operatorname{codim}(M) < n\},$$

and

$$d_n(S) := \inf\{\|Q_N^F S\|: \dim(N) < n\},$$

respectively, where J_M^E denotes the embedding map from M into E and Q_N^F the canonical map from F onto F/N. We mention that $a := (a_n)$, $c := (c_n)$ and $d := (d_n)$ are additive and multiplicative s-number functions [15, 11)]. If $s \in \{a, c, d\}$, then the quasinormed operator ideals $[\mathscr{L}_{p,q}^{(s)}; L_{p,q}^{(s)}]$ are analogously defined as above (cf. [15, (14)].

In the sequel $\rho, \rho_0, \ldots,$ are positive constants which may depend on exponents p, q, u, v, but not on operators, Banach spaces, or the rank of operators.

1. Inequalities of Lewis-Type

Let $[\mathfrak{A};\alpha]$ and $[\mathfrak{B};\beta]$ be quasinormed operator ideals. If there are constants $\lambda \geqslant 0$ and $\rho \geqslant 1$ such that (*) $\alpha(S) \leqslant \rho n^{\lambda}\beta(S)$ for $S \in \mathscr{L}$ with $\operatorname{rank}(S) \leqslant n$ and $n = 1, 2,\ldots$, then (*) is called an inequality of Lewis-type. For examples we refer to [2, 8–10, 17, 19, 21]. We start our considerations with a useful lemma.

Lemma 1. *Let E_n be an n-dimensional Banach space and let I_n be the identity operator on E_n. If $0 < q < \infty$ and $0 < v \leqslant \infty$, then*

$$L_{q,v}^{(e)}(I_n) \leqslant \rho n^{1/q} \qquad \text{for} \quad n = 1, 2,\ldots.$$

Proof. From [15, (12.1.13)], we know that

$$e_k(I_n) \leqslant 4 \cdot 2^{-(k-1)/2n} \qquad \text{for} \quad k = 1, 2,\ldots.$$

First we assume $0 < v \leqslant q < \infty$. Then

$$\begin{aligned}\sum_1^{\infty} k^{v/q-1} e_k^v(I_n) &\leqslant \sum_1^{n} k^{v/q-1} + 4^v \sum_{n+1}^{\infty} k^{v/q-1}(2^{-v/2n})^{(k-1)}\\ &\leqslant \rho_0 n^{v/q} + \rho_1 n^{v/q-1} \sum_{n+1}^{\infty} (2^{-v/2n})^{(k-1)}\\ &\leqslant \rho_0 n^{v/q} + \rho_1 n^{v/q-1}(1-2^{-v/2n})^{-1}.\end{aligned}$$

Using $2^{v/2n} \geqslant 1 + v/2n$ we get

$$\sum_1^{\infty} k^{v/q-1} e_k^v(I_n) \leqslant \rho_0 n^{v/q} + \rho_2 n^{v/q} \leqslant \rho_3 n^{v/q}.$$

This implies the desired inequality

$$L_{q,v}^{(e)}(I_n) \leqslant \rho n^{1/q} \qquad \text{for} \quad 0 < v \leqslant q < \infty \text{ and } n = 1, 2,\ldots.$$

The remaining case where $q \leqslant v \leqslant \infty$ can be checked from $L_{q,v}^{(e)}(I_n) \leqslant \rho_{q,v} L_{q,q}^{(e)}(I_n)$ by applying the above inequality.

The following Lewis-type inequality complements that of Lemma 14 in [17].

Lemma 2. *Let $0 < q < p \leqslant \infty$ and $0 < u, \ v \leqslant \infty$. Then $L_{q,v}^{(e)}(S) \leqslant \rho n^{1/q-1/p} L_{p,u}^{(e)}(S)$ for $S \in \mathscr{L}$ with* $\operatorname{rank}(S) \leqslant n$ *and* $n = 1, 2,\ldots.$

Proof. Given an operator $S \in \mathscr{L}(E, F)$ with $\operatorname{rank}(S) \leqslant n$ we write

$$\begin{array}{ccc} E & \xrightarrow{S} & E \\ {\scriptstyle S_0}\downarrow & & \uparrow{\scriptstyle J} \\ S(E) & \xrightarrow{I} & S(E) \end{array}$$

where S_0 is the astriction of S, I is the identity operator on $S(E)$ and J the natural injection. Clearly, $\dim S(E) \leqslant n$. Let $0 < q < p \leqslant \infty$, $1/r = 1/q - 1/p$. The product formula for $\mathscr{L}^{(e)}_{q,v}$ together with Lemma 1 yields

$$\begin{aligned} L^{(e)}_{q,v}(S) &\leqslant \rho_0 L^{(e)}_{r,v}(JI)\, L^{(e)}_{p,\infty}(S_0) \\ &\leqslant \rho_1 n^{1/r} L^{(e)}_{p,\infty}(S_0) \leqslant \rho_2 n^{1/q - 1/p} L^{(e)}_{p,u}(S_0). \end{aligned}$$

Now the desired inequality is a consequence of the injectivity of the entropy quasinorms $L^{(e)}_{p,u}$.

The next inequality of Lewis-type is the key for our later work.

LEMMA 3. *Let* $0 < q < p \leqslant \infty$ *and* $0 < u$, $v \leqslant \infty$. *Then* $L^{(e)}_{q,v}(S) \leqslant \rho n^{1/q - 1/p} L^{(a)}_{p,u}(S)$ *for* $S \in \mathscr{L}$ *with* $\operatorname{rank}(S) \leqslant n$ *and* $n = 1, 2, \ldots.$

Proof. Let $S \in \mathscr{L}(E, F)$ with $\operatorname{rank}(S) \leqslant n$ and put $N := [\log_2 n]$. By the definition of the approximation numbers we can find operators $S_j \in \mathscr{L}(E, F)$ with $\operatorname{rank}(S_j) < 2^j$ and

$$\|S - S_j\| \leqslant 2 a_{2^j}(S) \qquad \text{for} \quad j = 0, 1, \ldots, N + 1,$$

where $S_0 = 0$ and $S_{N+1} = S$. Put $D_j := S_j - S_{j-1}$, $j = 1, \ldots, N + 1$, then $\operatorname{rank}(D_j) < 2^{j+1}$ and $S = \sum_1^{N+1} D_j$. Using the factorization

$$\begin{array}{ccc} E & \xrightarrow{D_j} & E \\ {\scriptstyle D_j^0}\downarrow & & \uparrow{\scriptstyle J} \\ D_j(E) & \xrightarrow{I} & D_j(E) \end{array}$$

again with I the identity operator on $D_j(E)$, $\dim D_j(E) < 2^{j+1}$, and J the natural injection, we have by Lemma 1 and the fact that $\|D_j^0\| = \|D_j\|$,

$$L^{(e)}_{q,v}(D_j) \leqslant L^{(e)}_{q,v}(JI)\, \|D_j^0\| \leqslant \rho_0 2^{(j+1)/q} \|D_j\|.$$

From

$$\|D_j\| \leqslant \|S - S_j\| + \|S - S_{j-1}\| \leqslant 4 a_{2^{j-1}}(S)$$

it follows that

$$L^{(e)}_{q,v}(D_j) \leqslant \rho_1 2^{(j-1)/q} a_{2^{j-1}}(S).$$

Since $L_{q,v}^{(e)}$ is a (complete) quasinorm, there is an equivalent $r := r(q, v)$ norm (cf. [15, (6.2.5)]). If $0 < q < p \leqslant \infty$ and $n = 1, 2, \ldots$, then

$$\begin{aligned}
L_{q,v}^{(e)}(S) &= L_{q,v}^{(e)}\left(\sum_1^{N+1} D_j\right) \leqslant \rho_2 \left(\sum_1^{N+1} L_{q,v}^{(e)}(D_j)^r\right)^{1/r} \\
&\leqslant \rho_3 \left(\sum_1^{N+1} 2^{(j-1)r/q} a_{2^{j-1}}(S)^r\right)^{1/r} \\
&\leqslant \rho_3 \left(\sum_1^{N+1} 2^{(1/q-1/p)r(j-1)} (2^{(j-1)/p} a_{2^{j-1}}(S))^r\right)^{1/r} \\
&\leqslant \rho_3 \left(\sum_1^{N+1} 2^{(1/q-1/p)r(j-1)}\right)^{1/r} \sup_{1 \leqslant j \leqslant N+1} 2^{(j-1)/p} a_{2^{j-1}}(S) \\
&\leqslant \rho n^{1/q-1/p} \sup_{1 \leqslant j \leqslant n} j^{1/p} a_j(S) \\
&\leqslant \rho n^{1/q-1/p} L_{p,\infty}^{(a)}(S) \leqslant \rho n^{1/q-1/p} L_{p,u}^{(a)}(S).
\end{aligned}$$

2. Relationships between Entropy Numbers and Approximation Numbers

The main result of this section is that $\mathscr{L}_{p,q}^{(a)}(E, F) \subsetneqq \mathscr{L}_{p,q}^{(e)}(E, F)$ for all Banach spaces E and F. This yields an answer to a problem of Pietsch posed in [16; 5; 15, (14.3.12)]. In the process we have an extension of a famous result of Mitjagin originally formulated in terms of ε-entropy from Hilbert space to Banach spaces [12] (cf. also [15, (12.2.5)]).

Theorem 1. *Let $0 < p < \infty$ and $s \in \{a, c, d\}$. If $S \in \mathscr{L}(E, F)$, then*

$$\sup_{1 \leqslant k \leqslant n} k^{1/p} e_k(S) \leqslant \rho_p \sup_{1 \leqslant k \leqslant n} k^{1/p} s_k(S)$$

for $n = 1, 2, \ldots$.

Proof. Let $S \in \mathscr{L}(E, F)$ and let n be a natural number. By the definition of the approximation numbers we find an operator L with $\operatorname{rank}(L) < n$ and $\|S - L\| \leqslant 2a_n(S)$. Applying the additivity and monotonicity of the approximation numbers we get

$$a_k(L) \leqslant a_k(S) + \|L - S\| \leqslant a_k(S) + 2a_n(S) \leqslant 3a_k(S) \quad \text{for} \quad 1 \leqslant k \leqslant n.$$

also we have by the additivity of the entropy numbers

$$e_n(S) \leqslant \|S - L\| + e_n(L) \leqslant 2a_n(S) + e_n(L).$$

From Lemma 3 we obtain

$$n^{1/q} e_n(L) \leqslant \left(\sum_1^n e_k^q(L)\right)^{1/q} \leqslant \rho_{qp} n^{1/q - 1/p} \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(L)$$

for $0 < q < p < \infty$ and thus (with $q = p/2$)

$$\begin{aligned} n^{1/p} e_n(L) &\leqslant \rho_{p/2p} \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(L) \\ &\leqslant 3\rho_{p/2p} \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(S) \end{aligned}$$

for $n = 1, 2, \ldots$.

Combining the above inequalities we arrive at

$$n^{1/p} e_n(S) \leqslant (2 + 3\rho_{p/2p}) \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(S) \qquad \text{for} \quad n = 1, 2, \ldots.$$

This inequality immediately implies

$$\begin{aligned} l^{1/p} e_l(S) &\leqslant (2 + 3\rho_{p/2p}) \sup_{1 \leqslant k \leqslant l} k^{1/p} a_k(S) \\ &\leqslant (2 + 3\rho_{p/2p}) \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(S) \end{aligned}$$

for $l = 1, \ldots, n$ and therefore

$$\sup_{1 \leqslant k \leqslant n} k^{1/p} e_k(S) \leqslant (2 + 3\rho_{p/2p}) \sup_{1 \leqslant k \leqslant n} k^{1/p} a_k(S) \quad \text{for } n = 1, 2, \ldots.$$

Since the entropy numbers are injective and surjective [15, (12.1.8)], the preceding inequality is also valid for the Gelfand and Kolmogorov numbers.

THEOREM 2. *Let* $0 < p < \infty$ *and* $0 < q \leqslant \infty$. *If* $s \in \{a, c, d\}$, *then* $\mathscr{L}_{p,q}^{(s)}(E, F) \subseteq \mathscr{L}_{p,q}^{(e)}(E, F)$ *for all Banach spaces E and F.*

Proof. As an immediate consequence of Theorem 1 we have

$$\mathscr{L}_{p,\infty}^{(a)}(E, F) \subseteq \mathscr{L}_{p,\infty}^{(e)}(E, F) \qquad \text{for} \quad 0 < p < \infty.$$

Using this inclusion the general case can be checked by real interpolation: If $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$, and $1/p = (1 - \theta)/p_0 + \theta/p_1$, then

$$(\mathscr{L}_{p_0,q_0}^{(e)}(E, F), \mathscr{L}_{p_1,q_1}^{(e)}(E, F))_{\theta,q} \subseteq \mathscr{L}_{p,q}^{(e)}(E, F)$$

for $0 < q_0, q_1, q \leqslant \infty$.

The proof of this fact is the same one as that for Theorem 14 in [17]. Furthermore we have by a result of Peetre and Sparr [14] that

$$(\mathscr{L}^{(a)}_{p_0,q_0}(E,F), \mathscr{L}^{(a)}_{p_1,q_1}(E,F))_{\theta,q} = \mathscr{L}^{(a)}_{p,q}(E,F).$$

Now given an exponent p with $0<p<\infty$, we can find p_0, p_1 and θ such that $0<p_0<p<p_1<\infty$ and $1/p=(1-\theta)/p_0+\theta/p_1$. Applying the preceding interpolation formulas we obtain from $\mathscr{L}^{(a)}_{p_i,\infty}(E,F) \subseteqq \mathscr{L}^{(e)}_{p_i,\infty}(E,F)$, $i=0,1$, that

$$\begin{aligned}\mathscr{L}^{(a)}_{p,q}(E,F) &= (\mathscr{L}^{(a)}_{p_0,\infty}(E,F), \mathscr{L}^{(a)}_{p_1,\infty}(E,F))_{\theta,q}\\ &\subseteqq (\mathscr{L}^{(e)}_{p_0,\infty}(E,F), \mathscr{L}^{(e)}_{p_1,\infty}(E,F))_{\theta,q} \subseteqq \mathscr{L}^{(e)}_{p,q}(E,F)\end{aligned}$$

for $0<q\leqslant\infty$.

Since the entropy ideals $\mathscr{L}^{(e)}_{p,q}$ are injective and surjective [15, (14.3.5)], the preceding inclusion is also valid for the ideals $\mathscr{L}^{(c)}_{p,q}$ and $\mathscr{L}^{(d)}_{p,q}$.

The preceding theorems can be used to characterize diagonal operators between l_u-spaces as well as to embed maps between Besov spaces by their entropy numbers. For a detailed study of this we refer to [3, 4]. As a simple application of Theorem 2 we get the announced result of Mitjagin [12] (cf. also [15, (12.2.5)]). Let us mention that on the Hilbert space H the class $\mathscr{L}^{(a)}_{p,q}(H,H)$ is usually denoted by $\mathscr{S}_{p,q}$.

PROPOSITION 1. *Let $0<p<\infty$ and $0<q\leqslant\infty$. Then*

$$\mathscr{L}^{(e)}_{p,q}(H,H) = \mathscr{S}_{p,q}.$$

Proof. For $S\in\mathscr{L}(E,F)$ the nth Hilbert number $h_n(S)$ is defined by

$$h_n(S) := \sup\{a_n(BSX) : \|X\|\leqslant 1, \|B\|\leqslant 1\},$$

where $X\in\mathscr{L}(l_2,E)$ and $B\in\mathscr{L}(F,l_2)$. In [15, (12.3.1)] it is shown that $h_n(S)\leqslant 2e_n(S)$. This and Theorem 2 yield

$$\mathscr{L}^{(a)}_{p,q}(E,F) \subseteqq \mathscr{L}^{(e)}_{p,q}(E,F) \subseteqq \mathscr{L}^{(h)}_{p,q}(E,F).$$

Here $h:=(h_n)$ is again an s-number function in the sense of Pietsch [15, (11.4)]. From [15, (11.3.4)], we know that $\mathscr{L}^{(h)}_{p,q}(H,H)=\mathscr{S}_{p,q}$ which implies the desired assertion $\mathscr{L}^{(e)}_{p,q}(H,H)=\mathscr{S}_{p,q}$.

3. Distribution of Eigenvalues

Since we are interested in eigenvalues, all Banach spaces under considerations are assumed to be complex.

If $S \in \mathscr{L}(E, E)$ is a compact operator, then $(\lambda_n(S))$ denotes the sequence of all eigenvalues counted according to their algebraic multiplicities and ordered such that $|\lambda_1(S)| \geqslant |\lambda_2(S)| \geqslant \cdots \geqslant 0$. If S has less than n eigenvalues, then we put $\lambda_n(S) = 0$.

In the sequel we prove interesting inequalities involving eigenvalues and entropy numbers for compact operators acting on a Banach space.

In order to prove the main result of this section we need the following elementary fact (cf. [17]).

Lemma 4. *Let $S \in \mathscr{L}(E, E)$ be a compact operator and $\lambda_n(S) \neq 0$. Then there is an n-dimensional S-invariant subspace E_n of E such that the operator $S_n \in \mathscr{L}(E_n, E_n)$ induced by S has exactly the eigenvalues $\lambda_1(S),..., \lambda_n(S)$.*

The following interesting inequalities can be successfully applied to the study of eigenvalue problems of operators acting on Banach spaces. A weaker result has already been announced in [24].

Theorem 3. *Let $S \in \mathscr{L}(E, E)$ be a compact operator. Then*

$$|\lambda_n(S)| \leqslant (\sqrt{2})^{(k-1)/n} e_k(S)$$

for k, $n = 1, 2,...$.

Proof. If $\lambda_n(S) = 0$ the inequality is trivial, so we may assume $\lambda_n(S) \neq 0$.

By the preceding lemma there is an n-dimensional S-invariant subspace E_n of E such that S_n, the restriction of S to E_n, possesses exactly the eigenvalues $\lambda_1(S),..., \lambda_n(S)$.

Let J_n and P_n denote the natural injection from E_n into E and any spectral projection from E onto E_n, respectively. Then we have for S_n the factorization:

$$\begin{array}{ccc} E_n & \xrightarrow{S_n} & E_n \\ {\scriptstyle J_n}\big\downarrow & & \big\uparrow{\scriptstyle P_n} \\ E & \xrightarrow{S} & E \end{array}$$

Since $\lambda_n(S) \neq 0$, there exists the inverse operator S_n^{-1}. Clearly, the eigenvalues of S_n^{-1} are $\lambda_n^{-1}(S),..., \lambda_1^{-1}(S)$. Obviously, for the identity operator on an n-dimensional (complex) Banach space we have $I_n = P_n S^N J_n (S_n^{-1})^N$.

Using the fact $e_m(I_n) \geqslant (\sqrt{2})^{(m-1)/n}$ [15, (12.1.13)] and the well-known spectral radius formula

$$\lim_{N\to\infty} \|(S_n^{-1})^N\|^{1/N} = |\lambda_n(S)|^{-1}$$

the multiplicativity of the entropy numbers yields

$$\begin{aligned}(\sqrt{2})^{-N(k-1)/n} &\leqslant e_{N(k-1)+1}(I_n) = e_{N(k-1)+1}(P_n S^N J_n (S_n^{-1})^N) \\ &\leqslant \|P_n\|\, e_{N(k-1)+1}(S^N)\, \|J_n\|\, \|(S_n^{-1})^N\| \\ &\leqslant \|P_n\|\, e_k^N(S)\, \|(S_n^{-1})^N\|.\end{aligned}$$

Hence,

$$\|(S_n^{-1})^N\|^{-1/N} \leqslant (\sqrt{2})^{(k-1)/n)}\, \|P_n\|^{1/N}\, e_k(S)$$

for $N = 1, 2, \ldots$.

Letting $N \to \infty$ we obtain the desired conclusion

$$|\lambda_n(S)| \leqslant (\sqrt{2})^{(k-1)/n}\, e_k(S)$$

for $k, n = 1, 2, \ldots$.

As an immediate consequence of the preceding theorem we get

THEOREM 4. *Let $S \in \mathscr{L}(E, E)$ be a compact operator. Then*

$$|\lambda_n(S)| \leqslant (\sqrt{2})^{(n-1)/n}\, e_n(S)$$

for $n = 1, 2, \ldots$.

Notice that we have found for the first time pseudo-s-numbers which admit estimations for the single eigenvalues $\lambda_n(S)$ in terms of the single entropy numbers $e_n(S)$. All well-known s-numbers such as approximation, Gelfand, Kolmogorov, or Weyl numbers [17] do not possess this property. Moreover, Theorems 2 and 4 imply the earlier Weyl-type inequalities between eigenvalues and approximation, Gelfand, or Kolmogorov numbers found in [7].

In a forthcoming paper the result of Theorem 3 was slightly improved by Triebel and the author in the following way (cf. [6]):

Let $S \in \mathscr{L}(E, E)$ be a compact operator. Then

$$\left(\prod_1^n |\lambda_i(S)|\right)^{1/n} \leqslant (\sqrt{2})^{(k-1)/n}\, e_k(S)$$

for $k, n = 1, 2, \ldots$.

Finally, we give two typical situations in which Theorem 4 applies to operators acting on Lorentz sequence spaces as well as on Besov function spaces, complementing and improving earlier results of [7, 8] and the author.

PROPOSITION 2. *Let* $1 < p \leqslant q < \infty$ *and let* $S \in \mathscr{L}(l_{q,1}, l_{q,1})$ *be an operator admitting the factorization* $S = DS_0$, *where* $D \in \mathscr{L}(l_{p,\infty}, l_{q,1})$ *is a diagonal operator generated by a sequence* $(\sigma_k) \in l_{r,t}$, $0 < r < \infty$, $0 < t \leqslant \infty$, *and* $S_0 \in \mathscr{L}(l_{q,1}, l_{p,\infty})$. *Then*

$$(\lambda_k(S)) \in l_{s,t} \qquad \textit{for} \quad 1/s = 1/r + 1/p - 1/q.$$

Proof. By [3] we have $(e_n(D)) \in l_{s,t}$ for $1/s = 1/r + 1/p - 1/q$. Thus $(e_n(S)) \in l_{s,t}$ and by Theorem 4

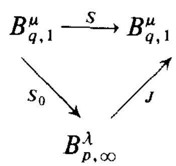
$$(\lambda_n(S)) \in l_{s,t} \qquad \text{for} \quad 1/s = 1/r + 1/p - 1/q.$$

Denote by $B^{\lambda}_{p,u}$, $1 \leqslant p$, $u \leqslant \infty$, $\infty < \lambda < \infty$, the Besov function spaces on a bounded interval. We have the following result.

PROPOSITION 3. *Let* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\lambda - \mu > 1/p - 1/q$. *If* $S \in \mathscr{L}(B^{\mu}_{q,1}, B^{\mu}_{q,1})$ *is an operator whose image is contained in* $B^{\lambda}_{p,\infty}$. *Then*

$$(\lambda_n(S)) \in l_{s,\infty} \qquad \textit{for} \quad 1/s = \lambda - \mu.$$

Proof. The operator S regarded as a map from $B^{\mu}_{q,1}$ into $B^{\lambda}_{p,\infty}$ has a closed graph and therefore S can be factorized in the following way:



where $S_0 \in \mathscr{L}(B^{\mu}_{q,1}, B^{\lambda}_{p,\infty})$ and J is the embedding map. By [4] we have

$$(e_n(J)) \in l_{s,\infty} \qquad \text{for} \quad 1/s = \lambda - \mu,$$

which implies $(e_n(S)) \in l_{s,\infty}$ and, thus, Theorem 4 yields

$$(\lambda_n(S)) \in l_{s,\infty} \qquad \text{for} \quad 1/s = \lambda - \mu.$$

Counterexamples exist to show that the results in the preceding propositions are optimal. In the case $1 \leqslant q \leqslant p \leqslant \infty$ eigenvalue distributions of such operators can be obtained by the technique of Weyl-numbers [17].

4. Inequalities of Bernstein–Jackson-Type

The inequalities in Section 2 concerning operators in Banach spaces can be interpreted as counterparts to the classical Bernstein and Jackson inequalities. It turns out that corresponding analogies exist between

(i) entropy numbers and the modulus of continuity,

(ii) approximation numbers and Bernstein numbers,

(iii) eigenvalues and Fourier coefficients.

Let $L_p^*[0, 1]$, $1 \leqslant p \leqslant \infty$, and $L_\infty^*[0, 1] := C^*[0, 1]$ for $p = \infty$ denote the spaces of p-summable and continous 1-periodic functions, respectively.

If $f \in L_p^*[0, 1]$, then the modulus of continuity is defined by

$$\omega^{(p)}(f, \delta) := \sup_{0 < |h| \leqslant \delta} \left(\int_0^1 |f(x+h) - f(x)|^p \, dx \right)^{1/p}$$

The nth Bernstein number $E_n^{(p)}(f)$, $f \in L_p^*[0, 1]$, is defined by

$$E_n^{(p)}(f) := \inf \|f - T\|_p, \qquad n = 0, 1, 2, \ldots,$$

where the infimum is taken over all trigonometrical polynomials T with degree$(T) < n$, $n = 1, 2, \ldots$, and where $T \equiv 0$ iff degree$(T) < 0$. Obviously, $E_0^{(p)}(f) = \|f\|_p$.

Bernstein inequalities

The Bernstein inequality for functions $f \in L_p^*[0, 1]$ says (cf. e.g., [11, 20])

$$\omega^{(p)}\left(f, \frac{1}{n}\right) \leqslant \frac{\delta}{n} \sum_1^n E_k^{(p)}(f) \qquad \text{for} \quad n = 1, 2, \ldots.$$

Theorem 1 implies an analogous (Bernstein) inequality for operators:

$$e_n(S) \leqslant \frac{\delta}{n} \sum_1^n a_k(S) \qquad \text{for} \quad n = 1, 2, \ldots.$$

Jackson inequalities

The Jackson inequalities for functions $f \in L_p^*[0, 1]$ says (cf. e.g., [11, 20])

$$E_n^{(p)}(f) \leqslant \rho \omega^{(p)}\left(f, \frac{1}{n}\right) \qquad \text{for} \quad n = 1, 2, \ldots.$$

By [15, (12.3.1)], we have an analogous (Jackson) inequality for operators:

$$h_n(S) \leqslant 2 e_n(S) \qquad \text{for} \quad n = 1, 2, \ldots.$$

If S acts between Hilbert spaces, then $h_n(S) = a_n(S)$ [15, (11.3.4)], and, thus,

$$a_n(S) \leqslant 2e_n(S) \qquad \text{for} \quad n = 1, 2, \ldots.$$

Notice that the analogies established above are not only formal. Indeed, if S_f is a convolution operator generated by a function f, then there are relationships between approximation numbers of S_f and the Bernstein numbers of f as well as relationships between the entropy numbers of S_f and the modulus of continuity of f. For this purpose let $f \in L_p^*[0, 1]$, $1 \leqslant p \leqslant \infty$, then the convolution operator S_f is defined by

$$S_f g := \int_0^1 f(s-t)\, g(t)\, dt.$$

The operator S_f can be considered as a map from $L_{p'}^*[0, 1]$ into $C^*[0, 1]$, $1/p' := 1 - 1/p$, cf. [23].

PROPOSITION 4. *Let $f \in L_p^*[0, 1]$, $1 \leqslant p \leqslant \infty$. Then for $S_f \in \mathscr{L}(L_{p'}^*[0, 1], C^*[0, 1])$ the inequalities*

$$a_1(S_f) = \|S_f\| \leqslant \|f\|_p = E_0^{(p)}(f),$$

$$a_{2n}(S_f) \leqslant E_n^{(p)}(f) \qquad \textit{for} \quad n = 1, 2, \ldots,$$

and

$$e_n(S_f) \leqslant \rho_r \left(n^{-1/r} \|f\|_p + \omega^{(p)}\left(f, \frac{1}{n}\right)\right)$$

for $0 < r \leqslant \infty$, $n = 1, 2, \ldots$, are valid.

Proof. Given $\varepsilon > 0$ and a natural number n. There is a polynomial T with degree$(T) < n$ and

$$\|f - T\|_p \leqslant (1+\varepsilon)\, E_n^{(p)}(f), \qquad f \in L_p^*[0, 1].$$

Since $\operatorname{rank}(S_T) < 2n$, we have

$$a_{2n}(S_f) \leqslant \|S_f - S_T\| \leqslant \|f - T\|_p \leqslant (1+\varepsilon)\, E_n^{(p)}(f).$$

This yields one of the desired inequalities.

Now, we turn to the other inequality. Without loss of generality we may assume $0 < r < 1$. In order to treat this case we need a consequence from Theorem 1: Let $0 < r < \infty$ and $S \in \mathscr{L}(E, F)$, then

$$e_n(S) \leqslant \rho_r n^{-1/r} \left(\sum_1^n a_k^r(S) \right)^{1/r} \qquad \text{for} \quad n = 1, 2, \ldots.$$

Using this, $a_{2n}(S_f) \leqslant E_n^{(p)}(f)$ and

$$E_n^{(p)}(f) \leqslant \rho\omega^{(p)}\left(f, \frac{1}{n}\right), \quad n = 1, 2, \ldots,$$

we obtain

$$\begin{aligned}
e_n(S_f) &\leqslant \rho_{r,0} n^{-1/r} \left(\sum_1^{2n} a_k^r(S_f)\right)^{1/r} \\
&\leqslant \rho_{r,0} n^{-1/r} \left(2a_1^r(S_f) + 2\sum_1^n a_{2k}^r(S_f)\right)^{1/r} \\
&\leqslant \rho_{r,1} n^{-1/r} \left(a_1(S_f) + \left(\sum_1^n a_{2k}^r(S_f)\right)^{1/r}\right) \\
&\leqslant \rho_{r,2} n^{-1/r} \left(\|f\|_p + \left(\sum_1^n E_k^{(p)}(f)^r\right)^{1/r}\right) \\
&\leqslant \rho_{r,3} n^{-1/r} \left(\|f\|_p + \left(\sum_1^n \omega^{(p)}\left(f, \frac{1}{k}\right)^r\right)^{1/r}\right) \qquad (*)
\end{aligned}$$

Now, we estimate the right-hand side. A well-known property of the modulus of continuity yields

$$\omega^{(p)}\left(f, \frac{1}{k}\right) \leqslant \left(\frac{n}{k} + 1\right) \omega^{(p)}\left(f, \frac{1}{n}\right).$$

Thus

$$\sum_1^n \omega^{(p)}\left(f, \frac{1}{k}\right)^r \leqslant \omega^{(p)}\left(f, \frac{1}{n}\right)^r \sum_1^n \left(\frac{n}{k} + 1\right)^r.$$

Since $0 < r < 1$, the sum on the right-hand side can be estimated by

$$\begin{aligned}
\sum_1^n \left(\frac{n}{k} + 1\right)^r &\leqslant \sum_1^n \left(\frac{2n}{k}\right)^r \leqslant (2n)^r \sum_1^n k^{-r} \\
&\leqslant \rho_{r,4} n^r n^{-r+1} = \rho_{r,4} n.
\end{aligned}$$

Hence

$$\left(\sum_1^n \omega^{(p)}\left(f, \frac{1}{k}\right)^r\right)^{1/r} \leqslant \rho_{r,5} n^{1/r} \omega^{(p)}\left(f, \frac{1}{n}\right). \qquad (**)$$

Combining the estimates (*) and (**) we obtain the desired conclusion

$$e_n(S_f) \leqslant \rho_r \left(n^{-1/r} \|f\|_p + \omega^{(p)} \left(f, \frac{1}{n} \right) \right)$$

for $0 < r < 1$, $n = 1, 2, \ldots$.

PROPOSITION 5. *Let* $f \in L_p^*[0, 1]$, $1 \leqslant p \leqslant \infty$, $0 < s$, $t \leqslant \infty$. *Then for* $S_f \in \mathscr{L}(L_{p'}^*[0, 1], C^*[0, 1])$ *the following conclusions are valid*:

$$(E_n^{(p)}(f)) \in l_{s,t} \text{ implies } S_f \in \mathscr{L}_{s,t}^{(a)}(L_{p'}^*[0, 1], C^*[0, 1])$$

and

$$\left(\omega^{(p)} \left(f, \frac{1}{n} \right) \right) \in l_{s,t} \text{ implies } S_f \in \mathscr{L}_{s,t}^{(e)}(L_{p'}^*[0, 1], C^*[0, 1]).$$

Proof. The first conclusion can be easy checked by Proposition 4. In order to treat the second one we choose a number r such that $0 < r < s$. Then $(n^{-1/r} \|f\|_p) \in l_{s,t}$ and by our assumption we have $(n^{-1/r} \|f\|_p + \omega^{(p)}(f, 1/n)) \in l_{s,t}$.

Now, Proposition 4 implies $(e_n(S_f)) \in l_{s,t}$ which yields the desired conclusion.

Finally, let $\lambda_n(f)$, $n = 0, \pm 1, \pm 2, \ldots$, be the (complex) Fourier coefficients of $f \in L_p^*[0, 1]$. It is well known that the $\lambda_n(f)$ are the eigenvalues of the convolution operator $S_f \in \mathscr{L}(L_{p'}^*[0, 1], L_{p'}^*[0, 1])$.

Fourier coefficients and eigenvalues

For a function $f \in L_p^*[0, 1]$ we have

$$|\lambda_n(f)| \leqslant \rho_p \omega^{(p)} \left(f, \frac{1}{n} \right) \qquad \text{for} \quad n = \pm 1, \pm 2, \ldots$$

On the other hand by Theorem 4 we have an analogous inequality for operators:

$$|\lambda_n(S)| \leqslant (\sqrt{2})^{(n-1)/n} e_n(S) \qquad \text{for} \quad n = 1, 2, \ldots$$

5. Final Remarks and an Open Problem

1. Using the ε-entropy defined by

$$H(S,\varepsilon) := \log \inf \left\{ n\colon \text{there are } y_1,\ldots, y_n \in F \text{ with } S(U_E) \subseteq \bigcup_1^n \{ y_i + \varepsilon U_F \} \right\},$$

the entropy ideals $\mathscr{L}^{(e)}_{s,t}$ can be described by

$$\mathscr{L}^{(e)}_{s,t} = \left\{ S \in \mathscr{L}\colon \int_0^{\|S\|} H^{t/s}(S;\varepsilon)\, d\varepsilon^t < \infty \right\}$$

if $0 < s < \infty$, $0 < t < \infty$, and

$$\mathscr{L}^{(e)}_{s,t} = \left\{ S \in \mathscr{L}\colon \sup \varepsilon^s H(S;\varepsilon) < \infty \right\}$$

if $0 < s < \infty$, $t = \infty$.

2. As mentioned in Section 2, in [3, 4] the results of Theorems 1 and 2 have been successfully applied to characterize diagonal operators between Lorentz sequence spaces as well as of embedding maps between Besov function spaces. We were able to extend results of Mitjagin, Marcus, Oloff, Birman and M. Z. Solomjak, and Triebel, to previously unknown cases.

3. Now, we give a corollary to Theorem 3. Let (s_n) be a multiplicative pseudo-s-number function (cf. [15, (12)]). For any compact, or more generally, for any Riesz operator (cf. [15, (26.5)]), $S \in \mathscr{L}(E, E)$ let $e_k(S) \leqslant \rho_k s_k(S)$ with $\rho_k^{1/k} \to 1$ if $k \to \infty$ (e.g., $\rho_k \leqslant \rho k^\sigma$ with $\rho \geqslant 1$ and $\sigma > 0$). Then

$$|\lambda_n(S)| \leqslant (\sqrt{2})^{(k-1)/n} s_k(S) \qquad \text{for} \quad k, n = 1, 2,\ldots$$

or slightly stronger

$$\left(\prod_1^n |\lambda_j(S)| \right)^{1/n} \leqslant (\sqrt{2})^{(k-1)/n} s_k(S) \qquad \text{for} \quad k, n = 1, 2,\ldots.$$

The proof of these inequalities can be carried out in the same way as the proof of Theorem 3.

4. Finally, we turn to general multiplicative s-number functions (s_n) [15, (11)]. As already shown by the author (unpublished, 1977), for any compact, or more generally, for any Riesz operator, $S \in \mathscr{L}(E, E)$ the following kind of Weyl-type inequality is valid:

$$\sum_{1}^{n} |\lambda_k(S)|^p \leqslant \rho_p \log(n+1) \sum_{1}^{[n/\log(n+1)]} s_k^p(S), \quad n = 1, 2, \ldots.$$

However, the following problem is open.

Problem. Is it true that

$$(\lambda_k(S)) \in l_{p,q} \qquad \text{for} \quad S \in \mathscr{L}_{p,q}^{(s)}(E, E)$$

and $0 < p < \infty$, $0 < q \leqslant \infty$?

For all known multiplicative s-numbers such as approximation, Gelfand, Kolmogorov, or Weyl numbers [17] the problem has been answered in the affirmative.

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