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CONSTRUCTIVE APPROXIMATION

Nonlinear Approximation and (Deep) ReLU Networks

I. Daubechies[1] · R. DeVore[2] · S. Foucart[2] · B. Hanin[2,3,4] · G. Petrova[2]

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Abstract
This article is concerned with the approximation and expressive powers of deep neural networks. This is an active research area currently producing many interesting papers. The results most commonly found in the literature prove that neural networks approximate functions with classical smoothness to the same accuracy as classical linear methods of approximation, e.g., approximation by polynomials or by piecewise polynomials on prescribed partitions. However, approximation by neural networks depending on n parameters is a form of nonlinear approximation and as such should be compared with other nonlinear methods such as variable knot splines or n-term approximation from dictionaries. The performance of neural networks in targeted applications such as machine learning indicate that they actually possess even greater approximation power than these traditional methods of nonlinear approximation. The main results of this article prove that this is indeed the case. This is done by exhibiting large classes of functions which can be efficiently captured by neural networks where classical nonlinear methods fall short of the task. The present article purposefully limits itself to studying the approximation of univariate functions by ReLU networks. Many generalizations to functions of several variables and other activation functions can be envisioned. However, even in this simplest of settings considered here, a theory that completely quantifies the approximation power of neural networks is still lacking.

Keywords Neural networks · Rectified linear unit (ReLU) · Expressiveness · Approximation power

Mathematics Subject Classification 41A25 · 41A30 · 41A46 · 68T99 · 82C32 · 92B20

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Extended author information available on the last page of the article

1 Introduction

Neural networks produce structured parametric families of functions that have been studied and used for almost 70 years, going back to the work of Hebb in the late 1940's [15] and of Rosenblatt in the 1950's [25]. In the last several years, however, their popularity has surged as they have achieved state-of-the-art performance in a striking variety of machine learning problems, from computer vision [19] (e.g., self-driving cars) to natural language processing [34] (e.g., Google Translate) and to reinforcement learning (e.g., superhuman performance at Go [30,31]). Despite these empirical successes, even their proponents agree that neural networks are not yet well-understood and that a rigorous theory of how and why they work could lead to significant practical improvements [3,20].

An often cited theoretical feature of neural networks is that they produce universal function approximators [5,16] in the sense that, given any continuous target function f on a compact domain and a target accuracy $\epsilon > 0$, neural networks with enough judiciously chosen parameters give an approximation to f within an error of size ϵ. Their universal approximation capacity has been known since the 1980's, yet it is not the main reason why neural networks are so effective in practice. Indeed, many other families of functions are universal function approximators. For example, one can approximate a fixed univariate real-valued continuous target function $f : [0, 1] \to \mathbb{R}$ using Fourier expansions, wavelets, orthogonal polynomials, etc. [10]. All of these approximation methods are universal. Not only that, but in these more traditional settings, through the core results of Approximation Theory [7,10], we have a complete understanding of the properties of the target function f which determine how well it can be approximated given a budget for the number of parameters to be used. Such characterizations do not exist for neural network approximation, even in the simplest setting when the target function is univariate and the network's activation function is the **Re**ctified **L**inear **U**nit (ReLU).

The neural networks used in modern machine learning are distinguished from those popular in the 1980's/90's by an emphasis on using *deep* networks (as opposed to shallow networks with one hidden layer). If the universal approximation property were key to the impressive recent successes of neural networks, then the depth of the network would not matter since both shallow and deep networks are universal function approximators.

The present article focuses on the advantages of deep versus shallow architectures in neural networks. Our goal is to put mathematical rigor into the empirical observation that deep networks can approximate many interesting functions more efficiently, per parameter, than shallow networks (see [12,13,26,33,36,37] for a selection of rigorous results).

In recent years, there has been a number of interesting papers that address the approximation properties of deep neural networks. Most of them treat ReLU networks since the rectified linear unit is the activation function of preference in many applications, particularly for problems in computer vision. Let us mention, as a short list, some papers most related to our work. It is shown in [11] that deep ReLU networks can approximate functions of d variables as well as linear approximation by algebraic polynomials with a comparable number of parameters. This is done by using the fact

(proved by Yarotsky [36]) that power functions x^v can be approximated with exponential efficiency by deep ReLU networks. Yarotsky also showed that certain classes of classical smoothness (Lipschitz spaces) can be approximated with rates slightly better than that of classical linear methods [37]. The main advantage of deep neural networks is that they can output compositions of functions cheaply. This fact has been exploited by many authors (see e.g., [24], where this approach is formalized, and [2] where this property is used to compare deep network approximation with nonlinear shearlet approximation).

In the present paper, we address the approximation power of ReLU networks and, in particular, whether such networks are truly more powerful in approximation efficiency than the classical methods of approximation. Although most of our results generalize to the approximation of multivariate functions, we discuss only the univariate setting since this gives us the best chance for definitive results. Our main focus is on the advantages of depth, i.e., what advantages are present in deep networks that do not appear in shallow networks. We restrict ourselves to ReLU networks since they have the simplest structure and should be easiest to understand.

We emphasize that when discussing approximation efficiency, we assume that f is fully accessible and we ask how well f can be approximated by a neural network with n parameters. This is in contrast to problems of data fitting where, instead of full access to f, we only have some data observations about it. In the latter case, the approximation can only use the given data and its performance would depend on the amount and form of that data. Performance in data fitting is often formulated in a stochastic setting in which it is assumed that the data is randomly generated and both the observations and the gradient descent parameter updates are noisy. The data fitting problem, using a specific form of approximation like neural networks, has two components, commonly referred to as bias and variance. We are concentrating on the bias component. It plays a fundamental role not only in data fitting but also in any numerical procedures based on neural network approximation.

Given two integers $W \geq 2$ and $L \geq 1$, we let (precise definitions are given in the next section)

$$\Upsilon^{W,L} := \{S : \mathbb{R} \to \mathbb{R},\ \ S \text{ is produced by a ReLU network of width } W \text{ and depth } L\}, \tag{1}$$

and denote by $n(W, L)$ the number of its parameters. We fix W and study the approximation families $\Upsilon^{W,L}$ when the number of layers L is allowed to vary. Our interest is in understanding why taking L large, i.e., why using deep networks is beneficial. One way to investigate the approximation power of $\Upsilon^{W,L}$ is to first compare it to known nonlinear approximation families with essentially the same number of degrees of freedom. Since every element in $\Upsilon^{W,L}$ is a **C**ontinuous **P**iece**w**ise **L**inear (CPwL) function, the classical approximation family closest to $\Upsilon^{W,L}$ is the nonlinear set

$$\Sigma_n := \{S : \mathbb{R} \to \mathbb{R},\ \ S \text{ is a CPwL function with at most } n \text{ distinct breakpoints in } (0, 1)\}.$$

The elements of Σ_n are also called free knot linear splines. We place the restriction that the breakpoints are in $(0, 1)$ because we are concerned with approximation on the interval $[0, 1]$.

When $n \asymp n(W, L)$, the sets Σ_n and $\Upsilon^{W,L}$ have comparable complexity in terms of parameters needed to describe them, since the elements in Σ_n are determined by $2n + 2$ parameters. This comparison also probes the expressive power[1] of depth for ReLU networks because Σ_W is (essentially) the same as the one-layer ReLU network $\Upsilon^{W,1}$, see (4).

Several interesting results [6,22,33] show that, for arbitrarily large $k \geq 1$ and $n = n(W, L)$ sufficiently large,

$$\Upsilon^{W,L} \setminus \Sigma_{n^k} \neq \emptyset, \tag{2}$$

cf e.g., [33, Theorem 1.2]. This means that sufficiently deep ReLU networks with n parameters can output certain CPwL functions whose number of breakpoints exceeds any power of n (the increase of the network depth is necessary as k grows). The reason for (2) is that composing two CPwL functions can increase the number of breakpoints, allowing networks with L layers of width W to create roughly W^L breakpoints for very special choices of weights and biases [33]. By choosing to use the available n parameters in a deep rather than shallow network, one can thus produce functions with many more breakpoints than parameters, albeit these functions have a very special structure.

The first natural question to answer in comparing Σ_n with $\Upsilon^{W,L}$ is whether, for every fixed $W \geq 2$, each function $S \in \Sigma_n$ is in a corresponding set $\Upsilon^{W,L}$ with $n(W, L) \asymp n$, i.e., with a comparable number of parameters. This would guarantee we do not lose anything in terms of expressive power when considering deep networks with fixed width W over shallow networks with fixed depth L. One of our results, Theorem 3.1, gives a resolution to this question and shows that up to a constant multiplicative factor, fixed-width ReLU networks depending on n parameters are at least as expressive as the free knot linear splines Σ_n. In other words, deep ReLU networks retain all of the approximation power of free knot linear splines but also add something since they can create functions which are far from being in Σ_n. We want to understand the new functions being created and how they can assist us in approximation and thus in data fitting. In this direction, we showcase in Sects. 5 and 6 two classes of functions easily produced by ReLU networks, one consisting of self-similar functions and the other emulating trigonometric functions. Appending these classes to Σ_n naturally provides a powerful dictionary for nonlinear approximation. Similar observation was announced in [12], where the authors noticed that highly oscillatory textures and the Weierstrass function can be exponentially well approximated by sparse ReLU networks.

What types of results could effectively explain the increased approximation power of deep networks as compared with other forms of approximation? One possibility is to exhibit classes K of functions on which the decay rate of approximation error for neural networks is better than for other methods (linear or nonlinear) while depending on the same number of parameters. On this point, let us mention that by now there are several theorems in the literature [2,4,23,28], which show that neural networks perform as well as certain classical methods such as polynomials, wavelets, and shearlets (but they do not show that neural networks perform any better than these methods), or

[1] By expressivity of a neural network, we mean the collection of functions the network outputs.

optimally represent certain function classes in terms of Kolmogorov rate-distortion theory, [2,12].

We seek more convincing results providing compact classes K that are subsets of Banach spaces X on which neural networks perform significantly better than other methods of approximation. In this direction, we mention at the outset that such sets K cannot be described by classical smoothness (such as Lipschitz, Sobolev, or Besov regularity) because for classical smoothness classes K, there are known lower bounds on the performance for any methods of approximation (linear or nonlinear). These lower bounds are provided by concepts such as entropy and widths. However, let us point out that there is an interesting little twist here that allows deep neural networks to give a slight improvement over classical approximation methods for certain Lipschitz, Sobolev, and Besov classes (see Theorems 7.3 and 7.4). This improvement is possible when the selection of parameters used in the approximation is allowed to be unstable.

Our results on the expressive power of depth describe certain classes of functions that can be approximated significantly better by $\Upsilon^{W,L}$ than by Σ_n when $n(W, L)$ is comparable to n, see Sect. 7.3. The construction of these new classes of functions exploits the fact that when S and T are functions in Σ_n, their composition $S \circ T$ can be produced by fixed-width ReLU networks depending on a number of parameters comparable to n. This composition property allows one to construct broad classes of functions, based on self similarity, whose approximation error decays exponentially when using deep networks but only polynomially when using Σ_n (due to the utter failure of this composition property for Σ_n).

2 Preliminaries and Notation

To set some notation, recall the definition of the ReLU function applied to $x = (x_1, \dots, x_d) \in \mathbb{R}^d$:

$$\mathrm{ReLU}(x_1, \dots, x_d) = (\mathrm{ReLU}(x_1), \dots, \mathrm{ReLU}(x_d)) = (\max\{0, x_1\}, \dots, \max\{0, x_d\}).$$

Definition 2.1 A fully connected feed-forward ReLU network $\mathcal{N}$ with width W and depth L is a collection of weight matrices $M^{(0)}, \dots, M^{(L)}$ and bias vectors $b^{(0)}, \dots, b^{(L)}$. The matrices $M^{(\ell)}$, $\ell = 1, \dots, L-1$, are of size $W \times W$, whereas $M^{(0)}$ has size $W \times 1$, and $M^{(L)}$ has size $1 \times W$. The biases $b^{(\ell)}$ are vectors of size W if $\ell = 0, \dots, L-1$, and a scalar if $\ell = L$. Each such network $\mathcal{N}$ produces a univariate real-valued function

$$A^{(L)} \circ \mathrm{ReLU} \circ A^{(L-1)} \circ \cdots \circ \mathrm{ReLU} \circ A^{(0)}(x), \quad x \in \mathbb{R},$$

where

$$A^{(\ell)}(y) = M^{(\ell)} y + b^{(\ell)}, \quad \ell = 0, \dots, L.$$

We define $\Upsilon^{W,L}$ as the set of such functions resulting from all possible choices of weights and biases.

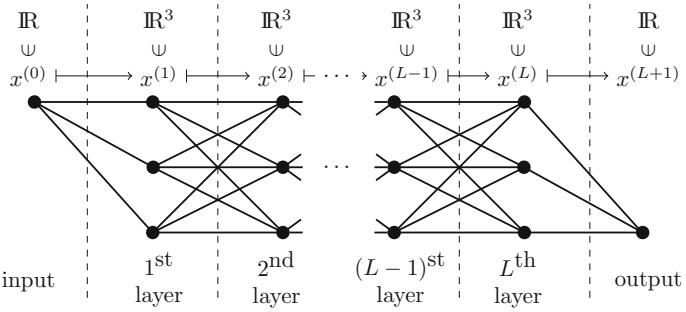

Fig. 1 Computation graph associated with a neural network with input/output dimension 1, width $W = 3$ and L hidden layers. The edges between layers $\ell - 1$ and ℓ are labeled by the entries of the weight matrix $M^{(\ell-1)}$. The j^{th} node (called a neuron) at layer ℓ computes the j^{th} component of $x^{(\ell)}$ by taking the dot product of the j^{th} row of $M^{(\ell-1)}$ with the entries of $x^{(\ell-1)}$ and adding it to the j^{th} entry of the vector $b^{(\ell-1)}$ of biases and applying ReLU to this quantity

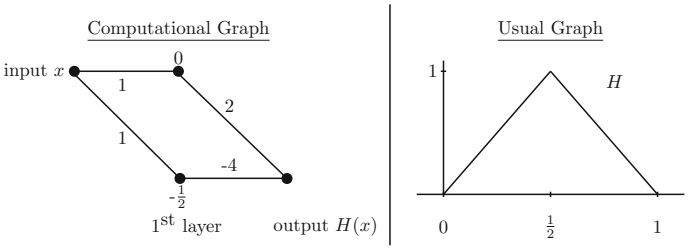

Fig. 2 Computation graph and usual graph associated with H

Every $S \in \Upsilon^{W,L}$ is a CPwL function on the whole real line. For each input $x := x^{(0)} \in \mathbb{R}$, the value $S(x^{(0)})$ of any $S \in \Upsilon^{W,L}$ is computed after the calculation of a series of intermediate vectors $x^{(\ell)} \in \mathbb{R}^W$, called vectors of activation at layer ℓ, $\ell = 1, \ldots, L$, before finally producing the output $x^{(L+1)} = M^{(L)}x^{(L)} + b^{(L)}$. The computations performed by such a network to produce an $S \in \Upsilon^{W,L}$ are shown schematically in Fig. 1.

For example, the *hat function* (also called *triangle function*) $H : [0, 1] \to \mathbb{R}$, defined as

$$H(x) = 2(x-0)_+ - 4\left(x - \frac{1}{2}\right)_+ = \begin{bmatrix} 2 & -4 \end{bmatrix} \mathrm{ReLU}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \right\}$$
$$= \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} < x \le 1, \end{cases} \tag{3}$$

belongs to $\Upsilon^{2,1}$, see Fig. 2.

For $L = 1$, each function in $\Upsilon^{W,1}$ is a CPwL function with at most W breakpoints determined by the nodes in the first layer. Conversely, any CPwL function with $(W-1)$ breakpoints interior to $[0, 1]$, when considered on the interval $[0, 1]$, is the restriction

of a function from $\Upsilon^{W,1}$ to that interval. Indeed, the elements $S \in \Sigma_{W-1}$ on $[0,1]$ can be represented as

$$ax + b + \sum_{j=1}^{W-1} m_j (x - \xi_j)_+ = \begin{bmatrix} a & m_1 & \dots & m_{W-1} \end{bmatrix} \mathrm{ReLU} \left\{ \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\xi_1 \\ \dots \\ -\xi_{W-1} \end{bmatrix} \right\} + b,$$

where $\xi_1, \dots, \xi_{W-1}$ are the interior breakpoints. In other words, as functions on $[0,1]$, we have

$$\Sigma_{W-1} \subset \Upsilon^{W,1} \subset \Sigma_W, \tag{4}$$

which means that for large W, the sets $\Upsilon^{W,1}$ and Σ_W are essentially the same. Therefore, neural networks with one hidden layer have the same approximation power as CPwL functions with the same number of parameters.

The number of parameters used to generate functions in $\Upsilon^{W,L}$ is

$$n(W,L) \;=\; W(W+1)L - (W-1)^2 + 2. \tag{5}$$

Not all counted parameters (the weights, i.e., entries of $M^{(\ell)}$, and biases, i.e., entries of $b^{(\ell)}$) are independent, since for instance some of the multipliers used in the transition $x^{(L)} \to x^{(L+1)}$ could have been absorbed in the preceding layer. We write

$$n(W,L) \asymp W^2 L$$

to indicate that $n(W,L)$ is comparable to $W^2 L$, in the sense that there are constants $c, C > 0$ such that $c\, W^2 L \le n(W,L) \le C\, W^2 L$—one could take $c = 1/2$ and $C = 2$ when $W \ge 2$ and $L \ge 2$.

3 ReLU Networks are at Least as Expressive as Free Knot Linear Splines

In this section, we fix $W \ge 4$, $L \ge 2$, and consider the set $\Upsilon^{W,L}$ defined in (1). Our goal is to prove that $\Sigma_n \subset \Upsilon^{W,L}$, where the number of its parameters $n(W,L) \le Cn$ for a certain fixed constant C. In order to formulate our exact result, we define $q := \lfloor \frac{W-2}{6} \rfloor$ when $W \ge 8$ and $q := 2$ for $4 \le W \le 7$.

Theorem 3.1 *Fix a width $W \ge 4$. For every $n \ge 1$, the set Σ_n of free knot linear splines with n breakpoints is contained in the set $\Upsilon^{W,L}$ of functions produced by width-W and depth-L ReLU networks, where*

$$L = \begin{cases} 2\left\lceil \frac{n}{q(W-2)} \right\rceil, & n \ge q(W-2), \\ 2, & n < q(W-2), \end{cases}$$

$$n(W,L) \le \begin{cases} Cn, & n \ge q(W-2), \\ W^2 + 4W + 1, & n < q(W-2), \end{cases}$$

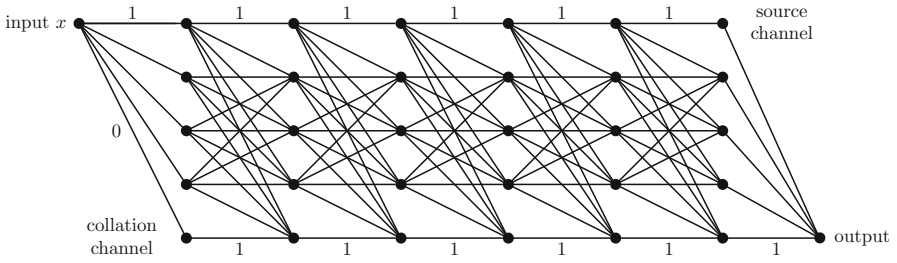

Fig. 3 Computation graph associated with $\overline{\Upsilon}^{5,6}$

with C an absolute constant. Here, $q := \lfloor \frac{W-2}{6} \rfloor$ *if* $W \geq 8$ *and* $q := 2$ *if* $4 \leq W \leq 7$.

Before giving the proof of Theorem 3.1 in Sect. 3.2, we first introduce in Sect. 3.1 some notation.

3.1 Special Neural Networks

Our main vehicle for proving Theorem 3.1 is the construction of a special neural network, whose output $\overline{\Upsilon}^{W,L}$ is subset of the output $\Upsilon^{W,L}$ of a RELU network with width W and depth L[2]. Given a width $W \geq 4$ and a depth $L \geq 2$, we focus on networks where a special role is reserved for two nodes in each hidden layer, see Fig. 3, which depicts these nodes as the first ("top") and at the last ("bottom") node of each hidden layer, respectively. The top neuron (first node), which is ReLU free, is used to simply copy the input x. The concatenation of all these top nodes can be viewed as a special "channel" (a term borrowed from the electrical engineering filter-bank literature) that skips computation altogether and just carries x forward. We call this the *source channel* (SC). The bottom neuron (last node) in each layer, which is also ReLU free, is used to collect intermediate results. We call the concatenation of all these bottom nodes the *collation channel* (CC). This channel never feeds forward into subsequent calculations, it only accepts previous calculations. The rest of the channels are *computational channels* (CmC). They consist of neurons (nodes), called computational nodes, that are equipped with the ReLU function, applied to the input and bias of that node. The fact that a special role is reserved for two channels enforces the natural restriction $W \geq 4$, since we need at least two computational channels. We call these networks (with SC and CC) *special* neural networks, for which we introduce a special notation, featuring a top and a bottom horizontal line to represent the SC and CC, respectively. Namely, we set

$$\overline{\Upsilon}^{W,L} = \{S : [0,1] \to \mathbb{R}, \ \ S \text{ is produced by a special network of width } W \text{ and depth } L\}.$$

We feel that these more structured networks are not only useful in proving results on approximation but may be useful in applications such as data fitting. In practice,

[2] Technically, the special networks differ from the usual ReLU networks because they contain ReLU-free neurons, but the set of functions $\overline{\Upsilon}^{W,L}$ produced by them is always contained in the standard ReLU network output $\Upsilon^{W,L}$, see Remark 3.1.

the designation of the first row as a SC and the last row as a CC amounts to having matrices $M^{(\ell)}$ and vectors $b^{(\ell)}$ of the form

$$M^{(0)} = \begin{bmatrix} 1 & m_2^{(0)} & \dots & m_{W-1}^{(0)} & 0 \end{bmatrix}^\top, \quad b^{(0)} = \begin{bmatrix} 0 & b_2^{(0)} & \dots & b_{W-1}^{(0)} & 0 \end{bmatrix}^\top,$$

$$M^{(\ell)} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ m_{2,1}^{(\ell)} & m_{2,2}^{(\ell)} & \dots & m_{2,W-1}^{(\ell)} & 0 \\ m_{3,1}^{(\ell)} & m_{3,2}^{(\ell)} & \dots & m_{3,W-1}^{(\ell)} & 0 \\ \dots & \dots & & & \\ m_{W-1,1}^{(\ell)} & m_{W-1,2}^{(\ell)} & \dots & m_{W-1,W-1}^{(\ell)} & 0 \\ m_{W,1}^{(\ell)} & m_{W,2}^{(\ell)} & \dots & m_{W,W-1}^{(\ell)} & 1 \end{bmatrix}, \quad b^{(\ell)} = \begin{bmatrix} 0 \\ b_2^{(\ell)} \\ b_3^{(\ell)} \\ \dots \\ b_{W-1}^{(\ell)} \\ b_W^{(\ell)} \end{bmatrix}, \quad \ell = 1, \dots, L-1, \quad (6)$$

and

$$M^{(L)} = \begin{bmatrix} m_1^{(L)} & \dots & m_{W-1}^{(L)} & 1 \end{bmatrix}, \quad b^{(L)} \in \mathbb{R}.$$

Remark 3.1 Note that since the SC and CC are ReLU-free, the width-W depth-L special networks do not form a subset of the set of width-W depth-L ReLU networks. However, in terms of sets of functions produced by these networks, the inclusion

$$\overline{\Upsilon}^{W,L} \subset \Upsilon^{W,L} \quad (7)$$

is valid. Indeed, given $\bar{S} \in \overline{\Upsilon}^{W,L}$, determined by the set of matrices and vectors $\{\bar{M}^{(\ell)}, \bar{b}^{(\ell)}\}$, $\ell = 0, \dots, L$, we will construct $\{M^{(\ell)}, b^{(\ell)}\}$, $\ell = 0, \dots, L$, such that $\bar{S}$ is also the output of a ReLU network with the latter matrices and vectors. First, notice that the input $x \in [0, 1]$, and therefore we have $x = \mathrm{ReLU}(x)$. Next, since the bottom neuron in the ℓ-th layer, $\ell = 1, \dots, L$, collects a function $\bar{S}^{(\ell)}(x)$ depending continuously on $x \in [0, 1]$, there is a constant C_ℓ such that $\bar{S}^{(\ell)}(x) + C_\ell \geq 0$ for all $x \in [0, 1]$. Hence, $\bar{S}^{(\ell)}(x) = \mathrm{ReLU}(\bar{S}^{(\ell)}(x) + C_\ell) - C_\ell$. Therefore, the ReLU network that produces $\bar{S}$ has the same matrices $M^{(\ell)} = \bar{M}^{(\ell)}$ and vectors $b^{(\ell)}$, $\ell = 1, \dots, L-1$, where

$$b_j^{(\ell)} = \bar{b}_j^{(\ell)}, \quad j = 1, \dots, W-1, \quad b_W^{(\ell)} = \bar{b}_W^{(\ell)} + C_\ell,$$

and $b^{(L)} = \bar{b}^{(L)} - \sum_{\ell=1}^{L-1} C_\ell$.

Proposition 3.2 *Special neural networks produce sets of CPwL functions that satisfy the following properties:*

(i) *For all W, L, Q,*

$$\overline{\Upsilon}^{W,L} + \overline{\Upsilon}^{W,Q} \subset \overline{\Upsilon}^{W,L+Q}. \quad (8)$$

(ii) *For $L < P$,*

$$\overline{\Upsilon}^{W,L} \subset \overline{\Upsilon}^{W,P}.$$

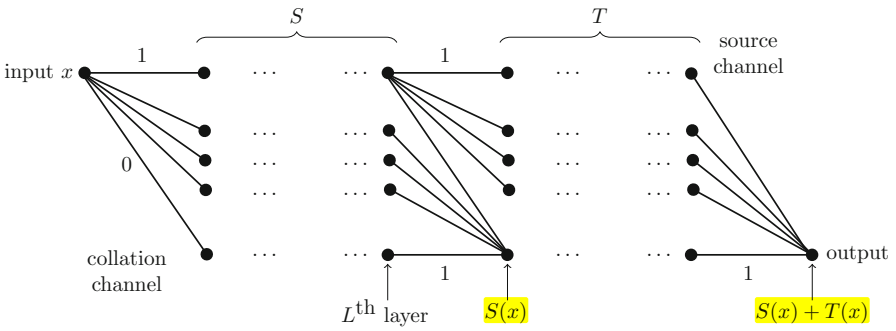

Fig. 4 Computational graph for summation

Proof To show (i), we first fix $S \in \overline{\Upsilon}^{W,L}$ and $T \in \overline{\Upsilon}^{W,Q}$ and use the following 'concatenation' of the special networks for S and T. The concatenated network has the same input and first L hidden layers as the network that produced S. Its $(L+1)$-st layer is the same as the first hidden layer of the network that produced T except that in the collation channel it places S rather than 0. The remainder of the concatenated network is the same as the remaining layers of the network producing T except that the collation channel is updated, see Fig. 4. The proof of (ii) follows the proof of (i) with $Q = P - L$ and $T \equiv 0$. □

3.2 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. Namely, we show that for any fixed width $W \geq 4$, any element T in Σ_n is the output of a special network with a number of parameters comparable to n.

Our constructive proof begins with Lemma 3.3, in which we create a special network with only 2 layers that generates a particular collection of CPwL functions, see (9). To describe this collection, we consider any positive integer N of the form $N := q(W-2)$, where $q := \lfloor (W-2)/6 \rfloor$. Since it is meaningful to have only cases when $q \geq 1$, we impose the restriction $W \geq 8$. In the "Appendix," we treat the remaining cases when $4 \leq W < 8$. Notice that N is small and so at this stage we are only showing how to construct CPwL functions with a few breakpoints.

Let $x_1 < \cdots < x_N \in (0,1)$ be any N given breakpoints in $(0,1)$ and choose x_0 and x_{N+1} to be any two additional points such that $0 \leq x_0 < x_1$ and $1 \geq x_{N+1} > x_N$. The set of all CPwL functions which vanish outside of $[x_0, x_{N+1}]$ and have breakpoints only at the $x_0, x_1, \ldots, , x_N, x_{N+1}$ is denoted by

$$\mathcal{S} := \mathcal{S}(x_0, \ldots, x_{N+1}) \tag{9}$$

and is a linear space of dimension N. We create a basis for $\mathcal{S}$ the following way. We denote by ξ_j, $j = 1, \ldots, (W-2)$, the points $\xi_j := x_{jq}$, which we call principal breakpoints and to each principal breakpoint ξ_j, we associate q basis functions $H_{i,j}$, $i = 1, \ldots, q$. Here, $H_{i,j}$, see Fig. 5, is a hat function supported on

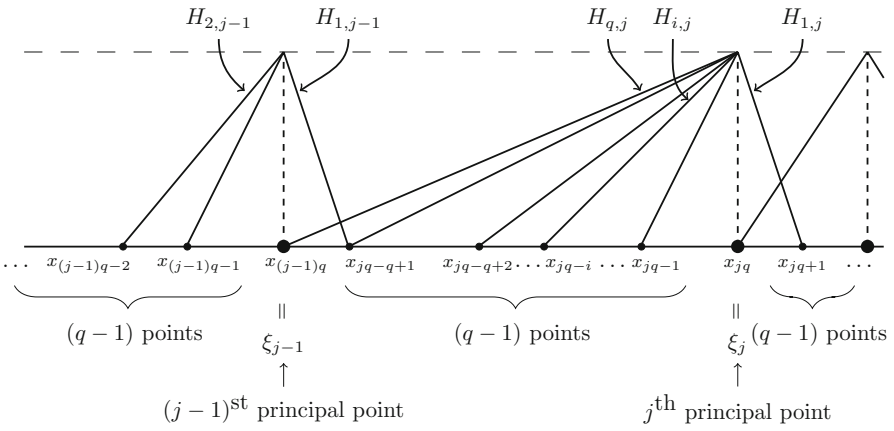

Fig. 5 Graphs of $H_{i,j}$

$I_{i,j} := [x_{jq-i,}, x_{jq+1}]$ which takes the value 0 at the endpoints of this interval, the value one at ξ_j and is linear on each of the two intervals $[x_{jq-i,}, x_{jq}]$ and $[x_{jq}, x_{jq+1}]$, that is

$$H_{i,j}(x) = \begin{cases} \frac{x-x_{jq-i}}{x_{jq}-x_{jq-i}}, & \text{if } x \in (x_{jq-i}, x_{jq}), \\ 0, & \text{if } x \notin I_{i,j}, \\ \frac{x-x_{jq+1}}{x_{jq}-x_{jq+1}}, & \text{if } x \in (x_{jq}, x_{jq+1}). \end{cases}$$

We rename these hat functions as ϕ_k, $k = 1, \ldots, N$, and order them in such a way that ϕ_k has leftmost breakpoint x_{k-1}, that is $\phi_{jq-i+1} = H_{i,j}$, $j = 1, \ldots, W-2$, $i = 1, \ldots, q$. We say ϕ_k is associated with ξ_j if ξ_j is the principal breakpoint where it is nonzero. We claim that these ϕ_k's are a basis for $\mathcal{S}$. Indeed, since there are N of them, we need to only check that they are linearly independent. If $\sum_{k=1}^{N} c_k \phi_k = 0$, then $c_1 = 0$ because ϕ_1 is the only one of these functions which is nonzero on $[x_0, x_1]$. We then move from left to right getting that each coefficient c_k is zero.

Lemma 3.3 *For any N breakpoints $x_1 < \cdots < x_N \in (0,1)$, $N := q(W-2)$, $q := \lfloor (W-2)/6 \rfloor$, $W \geq 8$, $\mathcal{S}(x_0, \ldots, x_{N+1}) \subset \overline{\Upsilon}^{W,2}$.*

Proof Consider $T \in \mathcal{S}(x_0, \ldots, x_{N+1})$, $T = \sum_{k=1}^{N} c_k \phi_k$, and determine its principal breakpoints $\xi_1, \ldots, \xi_{W-2}$ (every q-th point from the sequence $(x_1, x_2, \ldots, x_N)$ is a principal breakpoint). We next represent the set of indices $\Lambda = \{1, \ldots, N\}$ as a disjoint union of $K \leq 6q \leq W-2$ sets Λ_i,

$$\Lambda = \cup_{i=1}^{K} \Lambda_i,$$

where the Λ_i's have the following two properties:

- For any $\Lambda' \in \{\Lambda_1, \ldots, \Lambda_K\}$, all of the coefficients c_k with $k \in \Lambda'$ of T have the same sign.

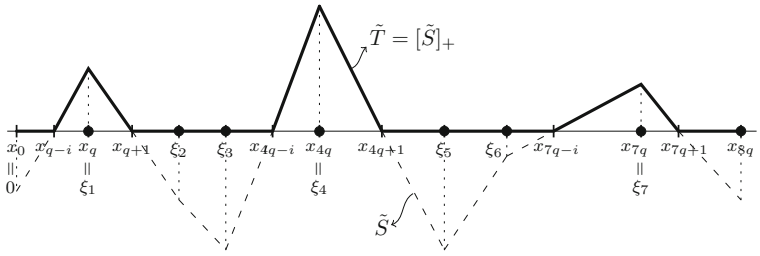

Fig. 6 A typical $\tilde{T}$ computed by a node in the second layer of $\overline{\Upsilon}^{W,2}$

- If $k, k' \in \Lambda'$, then the principal breakpoints ξ_j and $\xi_{j'}$ associated with $\phi_k, \phi_{k'}$, respectively, satisfy the separation property $|j - j'| \geq 3$.

We can find such a partition as follows. First, we divide $\Lambda = \Lambda_+ \cup \Lambda_-$ where for each $i \in \Lambda_+$, we have $c_i \geq 0$ and for each $i \in \Lambda_-$, we have $c_i < 0$. We then divide each of Λ_+ and Λ_- into at most $3q$ sets having the desired separation property. If $K < W - 2$, we set $\Lambda_{K+1} = \cdots = \Lambda_{W-2} = \emptyset$. It may also happen that some of the Λ_k's, $k \leq K$, are empty. In all cases for which $\Lambda_k = \emptyset$, we set $T_k = 0$, and write

$$T = \sum_{k=1}^{W-2} T_k, \quad T_k := \sum_{i \in \Lambda_k} c_i \phi_i, \quad k = 1, \ldots, W-2. \tag{10}$$

Notice that the $\phi_i, i \in \Lambda_k \neq \emptyset$, have disjoint supports and so $c_i = T_k(\xi_j)$ where ξ_j is the principal breakpoint associated with ϕ_i.

We next show that each of the T_k corresponding to a nonempty Λ_k is of the form $\pm[S_k(x)]_+$ for some linear combination S_k of $1, x, (x - \xi_1)_+, \ldots, (x - \xi_{W-2})_+$. Fix k and first consider the case where all of the c_i in Λ_k are nonnegative. We consider the CPwL function S_k which takes the value c_i at each principal breakpoint ξ_j associated with an $i \in \Lambda_k$. At the remaining principal breakpoints, we assign negative values to the $S_k(\xi_j)$'s. We choose these negative values so that for any $i \in \Lambda_k$, S_k vanishes at the leftmost and rightmost breakpoints of all ϕ_i with $i \in \Lambda_k$. This is possible because of the separation property. It follows that $[S_k(x)]_+ = T_k(x)$. A similar construction applies when all the coefficients in Λ_k are negative. In this case, $T_k = -[S_k]_+$ for the constructed S_k. We have suggested a particular strategy for defining the Λ_k's in "Appendix 9.1". A typical T_k, resulting from this strategy, which for the sake of simplicity we call $\tilde{T}$, is pictured in Fig. 6.

We can now describe the ReLU network that generates T. Since it is special, we focus only the computational channels. The computational nodes in the first layer are $(x - \xi_j)_+$, $j = 1, \ldots, W-2$, where the ξ_j's are the principal breakpoints. The computational nodes in the second layer are equal to the $[S_k]_+$ or 0. Because of (10), the target T is the output of this network with output layer weights ± 1 or 0. □

Remark 3.2 If we want to generate all spaces $\mathcal{S}(x_0, \ldots, x_{N_0+1})$ with $N_0 < N$ as outputs of a special network, we can artificially add $(N - N_0)$ distinct points in the interval (x_{N_0}, x_{N_0+1}) and view the elements in $\mathcal{S}(x_0, \ldots, x_{N_0+1})$ as CPwL with N

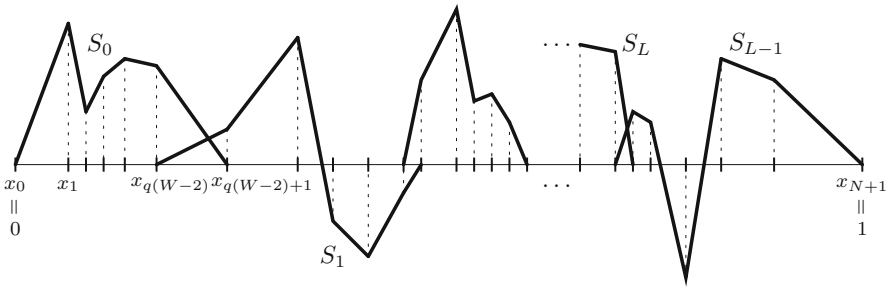

Fig. 7 Graphs of the functions S_j, $j = 0, \ldots, L-1$, from Lemma 3.4

breakpoints vanishing outside $[x_0, x_{N_0+1}]$, even though the last $N - N_0 + 1$ points are not really breakpoints, except possibly x_{N_0+1}.

Our next lemma shows how to carve up the target function $T \in \Sigma_n$ with a (possibly) large number of breakpoints into "bitesize" pieces that are handled by Lemma 3.3.

Lemma 3.4 *If $T \in \Sigma_N$ is any CPwL function on $[0, 1]$ with $N = q(W-2)L$, $q := \lfloor \frac{W-2}{6} \rfloor$, $W \geq 8$, then T is the output of a special network $\overline{\Upsilon}^{W,2L}$ with at most $2L$ layers.*

Proof Let $x_1 < \cdots < x_N$ be the breakpoints of T in $(0, 1)$ and set $x_0 := 0$, $x_{N+1} := 1$. We define $\ell(x) := ax + b$ to be the linear function which interpolates T at the endpoints $0, 1$ and set $S := T - \ell$. We can write $S = S_0 + \cdots + S_{L-1}$, where $S_j \in \Sigma_N$ is the CPwL function which agrees with S at the points x_i, for all indices $i \in \{jq(W-2)+1, \ldots, (j+1)q(W-2)\}$ and is zero at all other breakpoints of T, see Fig. 7.

Clearly, see (9),

$$S_j \in \mathcal{S}(x_{jq(W-2)}, \ldots, x_{(j+1)q(W-2)+1}), \quad j = 0, \ldots, L-1,$$

and therefore, it follows from Lemma 3.3 that each $S_j \in \overline{\Upsilon}_j^{W,2}$. We concatenate the L networks that produce $S_j \in \overline{\Upsilon}_j^{W,2}$, $j = 0, \ldots, L-1$, as described in Proposition 3.2 and thereby produce S. In order to account for the linear term $\ell(x)$, we assign weight a and bias b to the output of the node of the source channel in the last layer of the concatenated network, see Fig. 8. □

Proof of Theorem 3.1: Now, we are ready to complete the proof of Theorem 3.1.
Case 1 We first consider the case when $W \geq 8$. Let $N_1 := q(W-2)$, where $q := \lfloor \frac{W-2}{6} \rfloor$. Given n, if $n \geq N_1$, we choose L minimal such that $N := q(W-2)L \geq n$, that is

$$L = L(n, W) := \left\lceil \frac{n}{q(W-2)} \right\rceil.$$

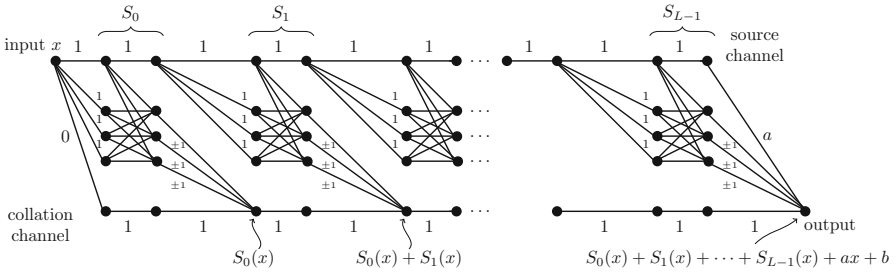

Fig. 8 Resulting network with $2L$ layers

Lemma 3.4 and inclusion (7) show that $\Sigma_N \subset \Upsilon^{W,2L}$ and therefore $\Sigma_n \subset \Sigma_N \subset \Upsilon^{W,2L}$. On the other hand, $L < \frac{n}{q(W-2)} + 1$. Using (5), we have that the number of parameters in $\Upsilon^{W,2L}$ is

$$\begin{aligned} n(W, 2L) &< 2W(W+1)\left(\frac{n}{q(W-2)} + 1\right) - (W-1)^2 + 2 \\ &= \frac{2W(W+1)}{q(W-2)} n + W^2 + 4W + 1. \end{aligned}$$

Optimizing over W shows that the maximum of $\frac{2W(W+1)}{q(W-2)}$ over integers $W \geq 8$ is achieved at $W = 13$ and $q = 1$, giving the value $\frac{364}{11} < 34$. Hence,

$$n(W, 2L) < 34n + W^2 + 4W + 1 < 34n + 27q(W-2) \leq 61n,$$

where we used that $W/13 \leq q$ and $q(W-2) = N_1 \leq n$.

On the other hand, if $n < N_1 := q(W-2)$, then Lemma 3.4 and inclusion (7) show that $\Sigma_n \subset \Sigma_{N_1} \subset \Upsilon^{W,2}$. Then, we have

$$n(W, 2) = W^2 + 4W + 1,$$

as desired.

Case 2 The proof of the case $4 \leq W \leq 7$ is given in "Appendix 9.2". □

Remark 3.3 A careful look at the proof of Theorem 3.1 gives in fact that

$$\Sigma_n \subset \overline{\Upsilon}^{W,L}$$

rather than the inclusion $\Sigma_n \subset \Upsilon^{W,L}$, with the same values of L and bounds on the number of parameters as described in Theorem 3.1.

Remark 3.4 We have not tried to optimize constants in the above theorem. If one counts the actual number of parameters used in $\Upsilon^{W,L}$ (rather than the parameters available), one obtains a much better constant. We know, in fact, that we can present other constructions (different than those given here) which provide a better constant in the statement of Theorem 3.1.

4 More About Standard and Special Networks

In this section, we discuss further properties of the sets $\Upsilon^{W,L}$ and $\overline{\Upsilon}^{W,L}$. We highlight in particular Theorem 4.1, which is a generalization of Theorem 3.1, and whose proof is deferred to "Appendix 9.3". Note that the conclusion of Theorem 3.1 depends on the ranges of the width W and the parameter n in Σ_n. To avoid excessive notation, we concentrate on only one of these ranges in the theorem below.

Theorem 4.1 *The following statement holds for compositions and sums of compositions of free knot linear splines:*

(i) *For functions $S_1 \in \Sigma_{n_1}, \dots, S_k \in \Sigma_{n_k}$ with $n_i \geq (W-2)\lfloor \frac{W-2}{6} \rfloor$, and $W \geq 8$, the composition*

$$S_k \circ \cdots \circ S_1 \in \Upsilon^{W,L}, \qquad L = 2\sum_{j=1}^{k} \left\lceil \frac{n_j}{\lfloor \frac{W-2}{6} \rfloor (W-2)} \right\rceil, \tag{11}$$

where the number of parameters describing $\Upsilon^{W,L}$ satisfies the bound

$$n(W,L) \leq 34 \sum_{j=1}^{k} n_j + 2k(W^2+W).$$

(ii) *For nonconstant functions $S_{i,j} \in \Sigma_{n_{i,j}}$, $i = 1, \dots, m$, $j = 1, \dots, \ell_i$, with $n_{i,j} \geq (W-4)\lfloor \frac{W-4}{6} \rfloor$, and $W \geq 10$, the sum of compositions satisfies*

$$\sum_{i=1}^{m} a_i S_{i,\ell_i} \circ \cdots \circ S_{i,1} \in \overline{\Upsilon}^{W,L} \subset \Upsilon^{W,L}, \tag{12}$$

where the number of parameters describing $\Upsilon^{W,L}$ satisfies the inequality

$$n(W,L) \leq 44 \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} n_{i,j} + 2W(W+1) \sum_{i=1}^{m} \ell_i.$$

Theorem 4.1 relies on some properties of standard and special networks. We state and prove below the ones that are explicitly needed in the remainder of the paper, starting with the following results.

Proposition 4.2 *Let $W \geq 2$. For any $\mathcal{Y}_1 \in \Upsilon^{W,L_1}, \dots, \mathcal{Y}_k \in \Upsilon^{W,L_k}$,*

(i) *The composition of the $\mathcal{Y}_i$ satisfies*

$$\mathcal{Y}_k \circ \cdots \circ \mathcal{Y}_1 \in \Upsilon^{W,L}, \qquad L = L_1 + \cdots + L_k; \tag{13}$$

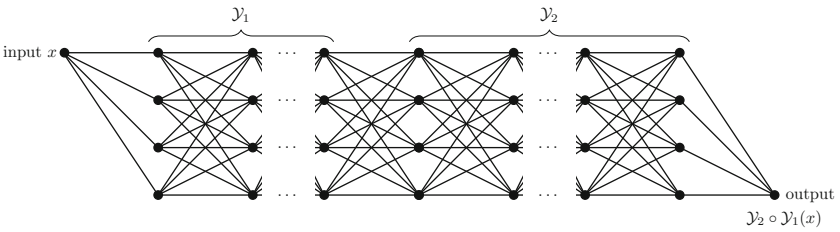

Fig. 9 Network computing $\mathcal{Y}_2 \circ \mathcal{Y}_1$

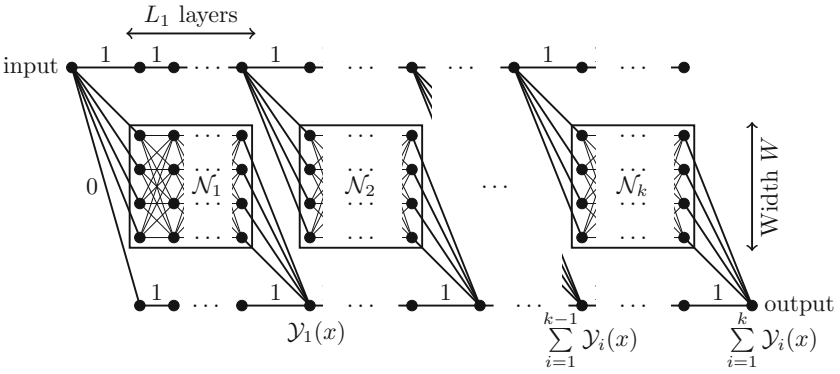

Fig. 10 Computational graph of the special network producing $\sum_{j=1}^{k} \mathcal{Y}_j$

(ii) *The sum of the $\mathcal{Y}_i$ satisfies*

$$\mathcal{Y}_1 + \cdots + \mathcal{Y}_k \in \overline{\Upsilon}^{W+2,L}, \qquad L = L_1 + \cdots + L_k; \tag{14}$$

(iii) *The sum of the $(\mathcal{Y}_i)_+ := \mathrm{ReLU} \circ \mathcal{Y}_i$ satisfies*

$$(\mathcal{Y}_1)_+ + \cdots + (\mathcal{Y}_k)_+ \in \overline{\Upsilon}^{W+2,L}, \qquad L = k + L_1 + \cdots + L_k. \tag{15}$$

Proof The argument is constructive. First, to prove (13), let $\mathcal{N}_j$ be the ReLU network with width W and depth L_j producing $\mathcal{Y}_j$. We concatenate the networks $\mathcal{N}_1, \cdots, \mathcal{N}_k$ as shown in Fig. 9 for the case of $\mathcal{Y}_2 \circ \mathcal{Y}_1$. The concatenated network has the same input and first L_1 hidden layers as the network $\mathcal{N}_1$. Its $(L_1 + 1)$-st layer is the same as the first hidden layer of the network $\mathcal{N}_2$. The weights between the L_1-st and $(L_1 + 1)$-st layer are the output weights of $\mathcal{Y}_1$, multiplied by the input weights for the first hidden layer of $\mathcal{Y}_2$. The remainder of the concatenated network is the same as the remaining layers of $\mathcal{N}_2$. Clearly, the resulting network will have $n = L_1 + \cdots + L_k$ hidden layers.

To show (14), we concatenate the networks $\mathcal{N}_1, \ldots, \mathcal{N}_k$ as shown in Fig. 4 by adding a source channel and a collation channel. The resulting network is a special network with width $W + 2$ and depth $L_1 + \cdots + L_k$.

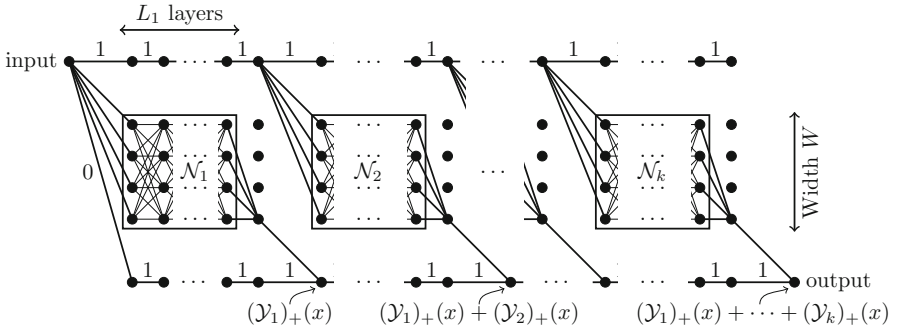

Fig. 11 Computational graph of the special network producing $\sum_{j=1}^{k}(\mathcal{Y}_j)_+$

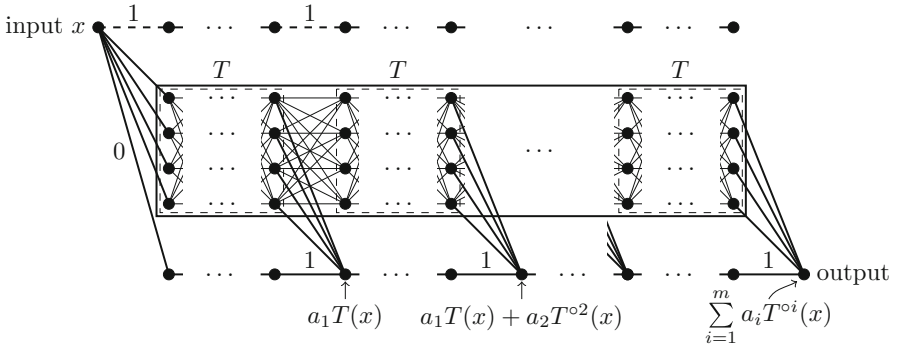

Fig. 12 Computational graph of the special network producing S

Finally, for (15), we concatenate the networks $\mathcal{N}_1, \ldots, \mathcal{N}_k$ by adding an extra layer after each $\mathcal{N}_j$ to perform the ReLU operation on its output, see Fig. 11. The rest of the construction is similar to the one for (14). □

The following two results will also be needed later. We use the notation $g^{\circ k}$, $k \geq 2$, to denote the function which results when g is composed with itself $k-1$times.

Proposition 4.3 *If $T \in \Upsilon^{\mathrm{w},L}$, $2 \leq \mathrm{w} \leq W$, then $S = \sum_{i=1}^{m} a_i T^{\circ i}$ can be produced by a special network with width $W+2$ and depth Lm, that is $S \in \overline{\Upsilon}^{W+2,Lm}$.*

Proof First, note that we have the inclusion $\Upsilon^{\mathrm{w},L} \subset \Upsilon^{W,L}$ for every $2 \leq \mathrm{w} \leq W$. We can always assign zero weights and biases to any selected nodes of the network producing $\Upsilon^{W,L}$, and therefore we can always assume that $T \in \Upsilon^{W,L}$. We adjust the network generating $T^{\circ m}$ encountered in the proof of (13). We augment it to a special network in such a way that after the computation of each of the $T^{\circ i}$, we place $a_i T^{\circ i}(x)$ into the collation channel, see Fig. 12. The source channel is not needed in this case, but we include it nonetheless since it will be used when creating the sum of S with another function. □

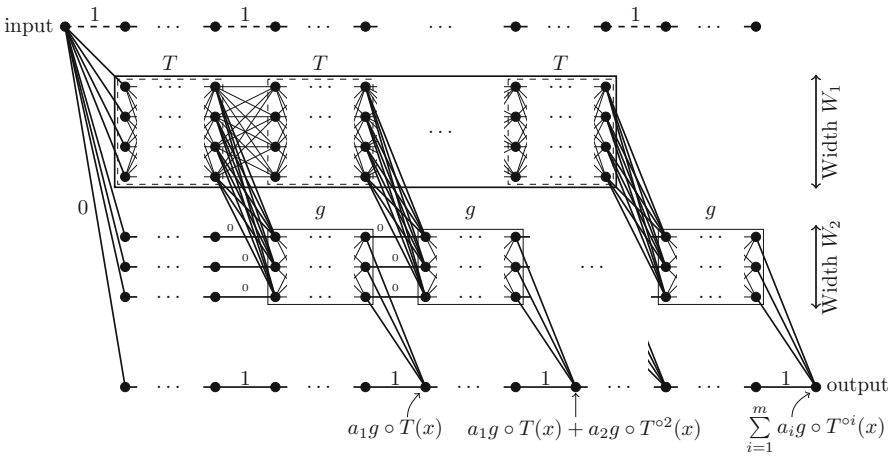

Fig. 13 Computational graph of the special network producing S_g

Proposition 4.4 *If $T \in \Upsilon^{W_1,\ell}$, $g \in \Upsilon^{W_2,\ell}$, and $W_1 + W_2 = W$, then $S_g = \sum_{i=1}^{m} a_i g \circ T^{\circ i}$ can be produced by a special network with width $W + 2$ and depth $\ell(m+1)$, i.e., $S_g \in \overline{\Upsilon}^{W+2,\ell(m+1)}$.*

Proof As before, we use the network of width W_1 generating $T^{\circ m}$. For the other W_2 channels, we use m copies of the network $\mathcal{G}$ producing g and combine them as shown in Fig. 13. After the computation of each of the $T^{\circ i}$, we place $T^{\circ i}(x)$ as an input in the i-th copy of $\mathcal{G}$ and put a_i times its output into the collation channel. Again, the source channel is not needed here but can be used at a later time. □

5 ReLU Networks Efficiently Produce Functions with Self Similarity

Having established that ReLU networks can output sums and compositions of CPwL functions, we show that they also can output CPwL functions with certain self-similar patterns. We formalize this structure below.

Let $0 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$ be a fixed set of breakpoints and let S be any element of $\mathcal{S}(\boldsymbol{\xi}) := \mathcal{S}(0, \xi_1, \ldots, \xi_k, 1)$. In particular, S vanishes outside of $[0, 1]$. We think of S as a *pattern*. It is easy and cheap for ReLU networks to replicate this pattern on many intervals. To describe this, let $\{J_1, \ldots, J_m\}$ denote a collection of m intervals contained in $[0, 1]$ whose interiors are pairwise disjoint. We order these intervals from left to right. We say that a CPwL function F is self similar with pattern $S \in \mathcal{S}(\boldsymbol{\xi})$ if

$$F(x) = \sum_{i=1}^{m} S(h_i(x - a_i)), \quad x \in [0, 1], \tag{16}$$

where $J_i = [a_i, b_i]$ and $h_i = |J_i|^{-1}$, $i = 1, \dots, m$. Thus, the function F consists of a dilated version of S on each of the m intervals J_i. It has roughly km breakpoints but is only described by $2(k+m)$ parameters. We show below that in order to produce such a function F, ReLU networks only need a number of parameters of the order $k+m$, and not km as would be naively inferred by regarding F as an element of Σ_{km}.

ReLU needs less params

Theorem 5.1 *Let $W \geq 8$. Any self-similar function F of the form (16) with $S \in \mathcal{S}(\boldsymbol{\xi}) \subset \Sigma_k$ belongs to $\overline{\Upsilon}^{W,L}$, for a suitable value of L that satisfies $n(W,L) \leq C_1(k+m) + C_2 W^2$ for some absolute constants $C_1, C_2 > 0$.*

Proof We start with the case when S is nonnegative and the intervals $J_i = [a_i, b_i]$ (not just their interiors) are disjoint. For each $i = 1, \dots, m$, we introduce a point c_i in the interval (b_i, a_{i+1}), where $a_{m+1} := 1$. We consider the hat function $\mathcal{H}_i$ which is zero outside $[a_i, c_i]$, equal to one at b_i, and linear on $[a_i, b_i]$ and $[b_i, c_i]$, as well as the hat function $\hat{\mathcal{H}}_i$ which is zero outside $[b_i, a_{i+1}]$, equal to one at c_i, and linear on $[b_i, c_i]$ and $[c_i, a_{i+1}]$. In the case when $b_m = 1$, we cannot construct $\mathcal{H}_m$ and $\hat{\mathcal{H}}_m$ as above, and instead set $\mathcal{H}_m(x) = \frac{1}{1-a_m}(x - a_m)_+$ and $\hat{\mathcal{H}}_m(x) = 0$. With $\hat{S}(x) := S(1-x)$, we claim that

$$F = \left(S \circ T - \hat{S} \circ \hat{T}\right)_+, \qquad \text{where} \quad T := \sum_{i=1}^{m} \mathcal{H}_i, \qquad \hat{T} := \sum_{i=1}^{m} \hat{\mathcal{H}}_i.$$

This can be easily verified by separating into the three cases $x \in [a_i, b_i]$, $x \in [b_i, c_i]$, and $x \in [c_i, a_{i+1}]$. According to Theorem 3.1, we have $S, \hat{S} \in \Upsilon^{W-4,L'}$ with either $W^2 L' \asymp n(W-4, L') \leq C'k$ or $L' = 2$, and $T, \hat{T} \in \Upsilon^{W-4,L''}$ with either $W^2 L'' \asymp n(W-4, L'') \leq C''m$ or $L'' = 2$. Then, by Proposition 4.2, we obtain that both $S \circ T, \hat{S} \circ \hat{T} \in \Upsilon^{W-4,L'+L''}$, that their difference $S \circ T - \hat{S} \circ \hat{T} \in \overline{\Upsilon}^{W-2,2(L'+L'')} \subset \Upsilon^{W-2,2(L'+L'')}$. At last, the function $F = \left(S \circ T - \hat{S} \circ \hat{T}\right)_+ \in \overline{\Upsilon}^{W,L'''}$, where $L''' = 1 + 2(L' + L'')$, and therefore $n(W, L''') \asymp W^2 L''' \leq c_1(k+m) + c_2 W^2$.

Now, in the case of a general pattern S with k breakpoints, we write $S = S_+ - S_-$, where S_+, S_- are nonnegative, vanish outside $[0,1]$, and have $k' \leq 2k$ breakpoints. We also decompose each sum (16) corresponding to S_+ and S_- into a sum over odd indices and a sum over even indices to guarantee disjointness of the underlying intervals. In this way, F is represented as a sum of the ReLU of four functions of the form $(S_i \circ T_i - \hat{S}_i \circ \hat{T}_i)$ each of them belonging to $\overline{\Upsilon}^{W-2,2(L'+L'')}$ and according to Proposition 4.2, it follows that $F \in \overline{\Upsilon}^{W,L}$, where $L = 4 + 8(L' + L'')$. Finally, a parameter count gives

$$n(W,L) \asymp W^2 L = 4W^2 + 8W^2(L' + L'') \leq C_1(k+m) + C_2 W^2,$$

where C_1 and C_2 are absolute constants and concludes the proof. □

Remark 5.1 The above argument also works if the condition $S \in \mathcal{S}(\boldsymbol{\xi}) \subset \Sigma_k$ is replaced by $S \in \Upsilon^{W-4,L}$, where $S(0) = S(1)$ and $n(W-4, L) \leq Ck$, with C being an absolute constant.

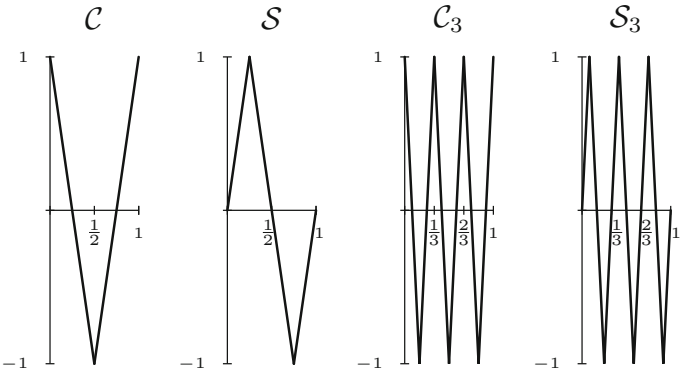

Fig. 14 Graphs of $\mathcal{C}$, $\mathcal{S}$, $\mathcal{C}_3$, and $\mathcal{S}_3$

6 ReLU Networks are at Least as Expressive as Fourier-like Sums

In this section, we show that ReLU networks can efficiently produce linear combinations of functions from a certain Riesz basis that emulates the trigonometric basis. The main point to emphasize here is that the linear combinations we consider can involve any of these basis functions not just the first consecutive ones. Such a linear combination consisting of n basis functions is commonly referred to as an n term approximation from a dictionary (a basis in our case). Approximation by such sums is a classic example of nonlinear approximation.

To describe the Riesz basis we have in mind, we consider the functions $\mathcal{C}, \mathcal{S} : [0, 1] \to \mathbb{R}$, given by

$$\mathcal{C}(x) := \begin{cases} 1 - 4x, & x \in [0, 1/2), \\ 4x - 3, & x \in [1/2, 1], \end{cases} \qquad \mathcal{S}(x) := \begin{cases} 4x, & x \in [0, 1/4), \\ 2 - 4x, & x \in [1/4, 3/4), \\ 4x - 4, & x \in [3/4, 1]. \end{cases}$$

Next, for each $k \geq 1$, we introduce $\mathcal{C}_k, \mathcal{S}_k : [0, 1] \to \mathbb{R}$, defined for any $x \in [0, 1]$ by

$$\mathcal{C}_k(x) := \mathcal{C}(kx - \lfloor kx \rfloor), \qquad \mathcal{S}_k(x) := \mathcal{S}(kx - \lfloor kx \rfloor).$$

Examples of representatives of this family of functions are depicted in Fig. 14. The system $\mathcal{F} := (\mathcal{C}_k, \mathcal{S}_k)_{k \geq 1}$ is an important example of a family of CPwL functions, since it forms a Riesz basis for $L_2^0[0, 1]$, the set of square integrable functions on $[0, 1]$ with zero mean. Namely, the following statement holds.

Proposition 6.1 *The system $(\mathcal{C}_k, \mathcal{S}_k)_{k \geq 1}$ is a Riesz basis for $L_2^0[0, 1]$, that is it spans $L_2^0[0, 1]$ and there are absolute constants $c, C > 0$ such that, for any two sequences $a, b \in \ell_2(\mathbb{N})$ of real numbers, we have*

$$c \sum_{k \geq 1} (a_k^2 + b_k^2) \leq \left\| \sum_{k \geq 1} (a_k \mathcal{C}_k + b_k \mathcal{S}_k) \right\|_{L_2[0,1]}^2 \leq C \sum_{k \geq 1} (a_k^2 + b_k^2). \tag{17}$$

Proof The proof of this statement is deferred to "Appendix 9.4" □

The following theorem shows how we can produce via ReLU networks $2k$-term linear combinations of elements from $\mathcal{F}$ with a good control on the depth L.

Theorem 6.2 *Let $W \geq 6$. For every set of indices $\Lambda \subset \mathbb{N}$, the set*

$$\mathcal{F}_\Lambda := \left\{ \sum_{j\in\Lambda} (a_j \mathcal{C}_j + b_j \mathcal{S}_j),\ a_j, b_j \in \mathbb{R},\ j \in \Lambda,\ |\Lambda| = k \right\} \subset \Upsilon^{W,L},$$

where

$$L = 2\left\lceil \frac{k}{\lfloor \frac{W-2}{4} \rfloor} \right\rceil (\lceil \log_2(\lambda) \rceil + 2), \qquad \text{with} \quad \lambda := \max \Lambda,$$

and the wights and biases in this network are bounded by $\max_{j\in\Lambda}\{|a_j|, |b_j|, 8\}$.

Proof With H denoting the hat function from Fig. 2, we observe that $H^{\circ m} = H \circ \cdots \circ H$ is a sawtooth function, see Fig. 15, i.e., a CPwL function taking alternatively the values 0 and 1 at its breakpoints $\ell 2^{-m}$, $\ell = 0, 1, \ldots, 2^m$. Note that the restriction of the function $(2^m x - \lfloor 2^m x \rfloor)$ on each interval $[\ell 2^{-m}, (\ell+1)2^{-m})$ is a linear function passing through $(\ell 2^{-m}, 0)$ with slope 2^m. Using the definitions of $\mathcal{C}$ and $H^{\circ m}$, one can easily see that

$$\mathcal{C}_{2^m}(x) = \mathcal{C}(2^m x - \lfloor 2^m x \rfloor) = \mathcal{C}(H^{\circ m}(x)).$$

Since H and $\mathcal{C} = 1 - 2H$ can both be produced by ReLU networks of width 2 and depth 1, it follows from (13) that $\mathcal{C}_{2^m} \in \Upsilon^{2,m+1}$, $m = 0, 1, \ldots$, and that all entries of the weight matrices and bias vectors of this ReLU network are bounded by 8, see (13) and Fig. 2.

Next, given an integer $j \geq 1$, we find the smallest $m \in \mathbb{N}_0$ with the property $j \leq 2^m$. In view of $\mathcal{C}_j(x) = \mathcal{C}_{2^m}(j2^{-m}x)$, $j \leq 2^m$, we also derive that $\mathcal{C}_j \in \Upsilon^{2,m+1} = \Upsilon^{2,\lceil \log_2 j \rceil + 1}$. Likewise, because $\mathcal{S}$ can be produced by a ReLU network of width 2 and depth 2 (by virtue of the identity $\mathcal{S}(x) = \mathcal{C}_2(x/2 + 3/8)$, $x \in [0,1]$), we can show that $\mathcal{S}_j \in \Upsilon^{2,m+2} = \Upsilon^{2,\lceil \log_2 j \rceil + 2}$. Thus, we have established that according to (14), for each $j \in \Lambda$,

$$a_j \mathcal{C}_j + b_j \mathcal{S}_j \in \overline{\Upsilon}^{4,2\lceil \log_2 j \rceil + 4} \subset \Upsilon^{4,2(\lceil \log_2 \lambda \rceil + 2)}, \quad \text{where} \quad \lambda := \max \Lambda, \tag{18}$$

and all entries of the weight matrices and bias vectors in this neural network are bounded by $\max\{|a_j|, |b_j|, 8\}$. Let us denote by $p := 2(\lceil \log_2 \lambda \rceil + 2)$. By stacking networks on top of each other, a sum of $\lfloor \frac{W-2}{4} \rfloor$ terms $a_j \mathcal{C}_j + b_j \mathcal{S}_j$ belongs to the set $\Upsilon^{4\lfloor \frac{W-2}{4} \rfloor, p} \subset \Upsilon^{W-2,p}$. Then, again by (14), a sum of $k \leq \lceil k/\lfloor \frac{W-2}{4} \rfloor \rceil \times \lfloor \frac{W-2}{4} \rfloor$ elements $a_j \mathcal{C}_j + b_j \mathcal{S}_j$ belongs to $\Upsilon^{W, \lceil k/\lfloor (W-2)/4 \rfloor \rceil p}$, as announced. □

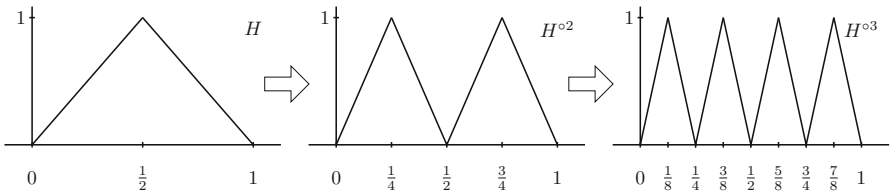

Fig. 15 Graphs of H, $H^{\circ 2}$, and $H^{\circ 3}$

Remark 6.1 Each element of the set $\mathcal{F}_\Lambda$ is described by $2k$ parameters, while the number of parameters $n(W, L)$ for the set $\Upsilon^{W,L}$ above has the order of $W^2 L \asymp Wk \log_2(\lambda)$. Ignoring the logarithmic factor, this is comparable with $2k$ only when the width W is viewed as an absolute constant.

We can take another approach and rather than stacking the networks producing $\mathcal{S}_j$ and $\mathcal{C}_j$ on the top of each other, concatenate them into a special network with width $W = 4$. This way, we will obtain that

$$\mathcal{F}_\Lambda \subset \Upsilon^{4,2k(\lceil \log_2(\lambda) \rceil + 2)}.$$

Remark 6.2 One can perform Fourier basis approximation via ReLU networks using the standard approach where, as it has been done for other bases such as wavelets, for example, one directly approximates each function from the (real) Fourier basis $(1, \cos(2\pi kx), \sin(2\pi kx))_{k\geq 1}$ using ReLU networks of constant width. An effort in this direction is Theorem IV.1 in [12], where using the fact that $\cos(ax)$ is an analytic function, the authors show that for every $\varepsilon \in (0, 1/2)$ and $a \in \mathbb{R}^+$, there is a ReLU network with width $W = 16$, depth $L = \mathcal{O}(\lceil \log(1/\varepsilon) \rceil^2 + \log a)$, and bounded weights and biases (by an absolute constant) that outputs $S \in \Upsilon^{16, \mathcal{O}(\lceil \log(1/\varepsilon) \rceil^2 + \log a)}$ such that

$$\|S - \cos(a\cdot)\|_{C[-1,1]} \leq \varepsilon.$$

In particular, for $\varepsilon = 2^{-n}$, $a = 2\pi j$, $S \in \Upsilon^{16, \mathcal{O}(n^2 + \log j)}$. One can now concatenate or stack these networks, as shown above, to produce a ReLU network whose output approximates a Fourier sum, $\sum_{j\in\Lambda}(a_j \cos(2\pi jx) + b_j \sin(2\pi jx))$ to a certain accuracy. While this is certainly a viable strategy to emulate Fourier basis approximation via ReLU neural networks, it is quite different from the approach above, where we use directly the highly oscillatory outputs $(\mathcal{C}_k, \mathcal{S}_k)_{k\geq 1}$ of ReLU networks with constant width and bounded weights and biases as our building blocks in the approximation. Even though one can show that $(\mathcal{C}_k, \mathcal{S}_k)_{k\geq 1}$ are linear splines with breakpoints the extrema of $(\cos(2\pi kx), \sin(2\pi kx))_{k\geq 1}$, respectively, that interpolate the latter functions at these points and the endpoints of $[0, 1]$, and thus can be viewed as approximations to the Fourier basis, they themselves are Riesz basis for $L_2^0[0, 1]$. The latter fact circumvents the role of the Fourier basis and makes any error estimates more or less straightforward.

7 Approximation by (Deep) Neural Networks

So far, we have seen in Sects. 3, 5, and 6 that ReLU networks can produce free knot linear splines, self-similar functions, and expansions in Fourier-like Riesz basis of CPwL functions using essentially the same number of parameters that are used to describe these sets. This implies that ReLU networks are at least as expressive as any of these sets of functions. In fact, they are at least as expressive as the union of these sets, which intuitively forms a powerful incoherent dictionary.

We are more interested in the approximation power of deep neural networks rather than their expressiveness. Of course, one expects these two concepts are closely related. The remainder of this paper aims at providing convincing results about the approximation power of ReLU networks that establishes their superiority over the existing and more traditional methods of approximation. We shall do so by concentrating on special networks $\overline{\Upsilon}^{W+2,L}$ with a fixed width $W+2$. We introduce the notation

$$\overline{\Upsilon}_m := \overline{\Upsilon}^{W+2,m} \subset \Upsilon^{W+2,m}, \qquad \text{when } m \geq 1,$$

and $\overline{\Upsilon}_0 := \{0\}$, and formally define the approximation family

$$\overline{\Upsilon} := (\overline{\Upsilon}_m)_{m\geq 0}.$$

The number of parameters determining the set $\overline{\Upsilon}_m$ is $n(W+2,m) \asymp W^2 m$, and in going further, we shall refer to them as *roughly* $W^2 m$. Recall that according to Proposition 3.2, this nonlinear family possesses the following favorable properties:

- Nestedness: $\overline{\Upsilon}_{m'} \subset \overline{\Upsilon}_m$ when $m' \leq m$;
- Summation property: $\overline{\Upsilon}_{m'} + \overline{\Upsilon}_m \subset \overline{\Upsilon}_{m'+m}$.

7.1 Nonlinear Approximation

Let X be any Banach space of (equivalence classes of) functions defined on $[0,1]$. The typical examples of X are the $L_p[0,1]$ spaces, $1 \leq p \leq \infty$, $C[0,1]$, Sobolev and Besov spaces. Our only stipulation on X, at this point, is that it should contain all continuous piecewise linear functions on $[0,1]$. Given $f \in X$, we define its approximation error when using deep neural networks to be

$$\sigma_m(f,\overline{\Upsilon})_X := \inf_{S\in\overline{\Upsilon}_m} \|f-S\|_X, \quad m \geq 0.$$

Since $\overline{\Upsilon}_0 := \{0\}$, we have $\sigma_0(f,\overline{\Upsilon})_X = \|f\|_X$. Given a compact subset $K \subset X$, we define the performance on K to be

$$\sigma_m(K,\overline{\Upsilon})_X := \sup_{f\in K} \sigma_m(f,\overline{\Upsilon})_X, \quad m \geq 0.$$

In other words, the approximation error on the class K is the worst error.

In a similar way, we define approximation error for other approximation families, in particular $\sigma_m(f,\Sigma)_X$ and $\sigma_m(K,\Sigma)_X$ when $\Sigma := (\Sigma_m)_{m\ge 0}$ is the family of continuous piecewise linear functions. We want to understand the decay rate of $(\sigma_m(f,\overline{\Upsilon})_X)_{m\ge 0}$ for individual functions f and of $(\sigma_m(K,\overline{\Upsilon})_X)_{m\ge 0}$ for compact classes $K \subset X$ and to compare them with the decay rate for other methods of approximation.

Another common way to understand the approximation power of a specific method of approximation such as neural networks is to characterize the following approximation classes. Given $r > 0$, the approximation class $\mathcal{A}^r(\overline{\Upsilon})_X$, $r > 0$, is defined as the set of all functions $f \in X$ for which

$$\|f\|_{\mathcal{A}^r(\overline{\Upsilon})_X} := \sup_{m\ge 0}(m+1)^r \sigma_m(f,\overline{\Upsilon})_X$$

is finite. While approximation rates other than $(m+1)^{-r}$ are also interesting, understanding the classes $\mathcal{A}^r$, $r > 0$, matches many applications in numerical analysis, statistics, and signal processing. The approximation spaces $\mathcal{A}^r(\overline{\Upsilon})_X$ are linear spaces. Indeed, if $f, g \in \mathcal{A}^r(\overline{\Upsilon})_X$ and $S_m, T_m \in \overline{\Upsilon}_m$ provide the approximants to f, g satisfying

$$\|f - S_m\|_X \le M(m+1)^{-r} \quad \text{and} \quad \|g - T_m\|_X \le M'(m+1)^{-r}, \quad m \ge 0,$$

then $S_m + T_m$ provides an approximant to $f + g$ satisfying

$$\|f+g-(S_m+T_m)\|_X \le (M+M')(m+1)^{-r} \le 2^r(M+M')(2m+1)^{-r}, \quad m \ge 0.$$

Since $S_m + T_m$ is in $\overline{\Upsilon}_{2m}$, we derive that $f + g \in \mathcal{A}^r(\overline{\Upsilon})_X$. We notice in passing that $\|\cdot\|_{\mathcal{A}^r(\overline{\Upsilon})_X}$ is a quasi-norm.

Approximation classes are defined for other methods of approximation in the same way as for neural networks. Thus, given a sequence $\mathcal{X} := (X_m)_{m\ge 0}$ of sets (linear or nonlinear), $X_0 := \{0\}$, we define $\mathcal{A}^r(\mathcal{X})_X$ as above with $\overline{\Upsilon}$ replaced by $\mathcal{X}$. The approximation spaces for all classical linear methods of approximation have been characterized for all $r > 0$ when $X = L_p[0,1]$, $1 \le p < \infty$, and $X = C[0,1]$. For example, these approximation classes are known for approximation by algebraic polynomials, by trigonometric polynomials, and by piecewise polynomials on an equispaced partition. Interestingly enough, these characterizations do not expose any advantage of one classical linear method over another. All of these approximation methods have essentially the same approximation classes. For example, the approximation classes $\mathcal{A}^r$ for approximation in $C[0,1]$ by piecewise constants on equispaced partition of $[0,1]$ are the Lip r spaces when $0 < r \le 1$. Here, the space Lip r is specified by the condition

$$|f(x) - f(y)| \le M|x-y|^r$$

and the smallest $M \ge 0$ for which this holds is by definition the semi-norm $|f|_{\mathrm{Lip}\, r}$. The space $\mathcal{A}^r$, $0 < r < 1$, remains the same if we use trigonometric polynomials of

degree m. The notion of Lipschitz spaces can be extended to $r > 1$ and then can be used to characterize approximation spaces $\mathcal{A}^r$ when $r > 1$. We do not go into more detail on approximation spaces for the classical linear spaces but we refer the reader to [10] for a complete description.

The situation changes dramatically when using nonlinear methods of approximation. There is typically a huge gain in favor of nonlinear approximation in the sense that their approximation classes are much larger than for linear approximation, and so it is easier for a function to have the approximation order $O(m^{-r})$. We give just one example, important for our discussion of neural networks, to pinpoint this difference. It is easy to see that any continuous function of bounded variation is in $\mathcal{A}^1(\Sigma)$. Namely, given such a target function f defined on $[0, 1]$ and with total variation one, we partition $[0, 1]$ into m intervals such that the variation of f on each of these intervals is $1/m$. Then, the CPwL function which interpolates f at the endpoints of these intervals is in Σ_m and approximates f with error at most $1/m$. Notice that such functions of bounded variation are far from being in Lip 1 because they can change values quite abruptly. This illustrates the central theme of nonlinear approximation that their approximation spaces are much larger than their linear counterparts. We refer the reader to [7] for an overview of nonlinear approximation.

7.2 Approximation of Classical Smoothness Spaces

Let us start this section by revisiting the statement of Theorem 3.1 and Remark 3.3, from where we derive that for $W \geq 4$

$$\Sigma_m \subset \Upsilon^{W, \lceil \frac{C}{W^2} \rceil m}, \quad m \geq q(W-2),$$

and

$$\Sigma_m \subset \overline{\Upsilon}^{W,2}, \quad 1 \leq m < q(W-2),$$

where $q = 2$ when $4 \leq W \leq 7$ and otherwise $q = \lfloor \frac{W-2}{6} \rfloor$. In addition, for any m we can embed $\Upsilon^{W,m} \subset \overline{\Upsilon}^{W+2,m} = \overline{\Upsilon}_m$ by adding a source and collation channel. Hence, in the view of the new notation, Theorem 3.1 can be restated the following way.

Theorem 7.1 *For $W \geq 4$ and $m \geq q(W-2)$, we have*

$$\Sigma_m \subset \overline{\Upsilon}_{\gamma m}, \quad \textit{where} \quad \gamma = \gamma(W) := \left\lceil \frac{C}{W^2} \right\rceil,$$

and thus for any $f \in C[0, 1]$

$$\sigma_{\gamma m}(f, \overline{\Upsilon})_{C[0,1]} \leq \sigma_m(f, \Sigma)_{C[0,1]}.$$

For $1 \le m < q(W-2)$,

$$\Sigma_m \subset \overline{\Upsilon}_2,$$

and thus for any $f \in C[0,1]$ *we have* $\sigma_2(f, \overline{\Upsilon})_{C[0,1]} \le \sigma_m(f, \Sigma)_{C[0,1]}$.

Therefore, all upper bounds for the error of best approximation $\sigma_m(f, \Sigma)_{C[0,1]}$ by the family Σ of free knot linear splines will hold for the error of best approximation $\sigma_{\gamma m}(f, \overline{\Upsilon})_{C[0,1]}$ by the family $\overline{\Upsilon}$. We refer the reader to the paper [7] for a detailed description of free knot spline approximation. To orient the discussion that follows, we mention a small set of results on the approximation of functions in $C[0,1]$. One of the best known results for approximation by CPwL functions is that any Lip 1 function can be approximated in the norm of $C[0,1]$ by a CPwL with n breakpoints to accuracy $\|f\|_{\text{Lip }1} n^{-1}$. This estimate can already be achieved by linear methods of approximation since the breakpoints in this result can be chosen equally spaced, and thus the approximation need not be chosen to nonlinearly dependent of f. On the other hand, exploiting the nonlinearity of Σ_n, one can show that the above rate of approximation $\mathcal{O}(n^{-1})$ holds with the much weaker assumption $f' \in L_1[0,1]$ in place of the Lip 1 assumption (which is equivalent to assuming $f' \in L_\infty[0,1]$). This fact does not hold when using CPwL functions with equally spaced breakpoints, and indeed, here one has to take full advantage of the nonlinear structure of Σ_n. It follows that these results hold equally well when using the approximation family Υ in place of Σ.

However, the question is can we say more when using deep neural networks in approximating these traditional smoothness classes? The answer is quite surprising and indicative. A series of results beginning with Yarotsky [36–38] and continuing in [17,29] show that the approximation rate of some classical smoothness classes, e.g., Lipschitz classes, is dramatically better when using deep networks. Results are now known for approximating the unit ball $U(W^s(L_p[0,1]^d))$, $s > 0$, of this Sobolev space with the approximation error measured in $L_p[0,1]^d$ for the same $p \in [1,\infty]$. We limit our discussion to the approximation in the $C[0,1]$ norm of classes like Lip 1 and refer the reader to the above references for the full spectrum of results. The first result in this direction [37] showed that for $W = 5$,

$$\sup_{f \in \text{Lip }1} \inf_{S \in \Upsilon^{W,n}} \|f - S\|_{C[0,1]} \le C \frac{|f|_{\text{Lip }1}}{n \ln n}. \tag{19}$$

This was a small but still surprising improvement over the optimal rate $\mathcal{O}(n^{-1})$ known for approximation by Σ_n. Later, for $W = 12$, this was improved to

$$\sup_{f \in \text{Lip }1} \inf_{S \in \Upsilon^{W,n}} \|f - S\|_{C[0,1]} \le C |f|_{\text{Lip }1} n^{-2}. \tag{20}$$

This is now known to be the optimal rate which cannot be improved. We shall return to discussing this result in more detail in Sects. 7.2.2 and 8. For now, we want to show how the original result (19) can be obtained from Theorem 5.1. The stronger results (20) require a different technique known as bit extraction.

7.2.1 The Space Lip α

We begin by discussing the Lip α spaces, $0 < \alpha \leq 1$. For this, we isolate a simple remark about the Kolmogorov entropy of the unit ball of Lip α. Let K_α be the set of functions $f : [0,1] \mapsto \mathbb{R}$ with $|f|_{\mathrm{Lip}\,\alpha} \leq 1$ vanishing at the endpoints 0 and 1.

Lemma 7.2 *For each $0 < \alpha \leq 1$ and for each integer $k \geq 2$, there are at most 3^k patterns $S_1, \ldots, S_{3^k}$ from $\mathcal{S}(\xi)$, $\xi = (0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1)$, such that whenever $g \in K_\alpha$, there is a $j \in \{1, \ldots, 3^k\}$ with*

$$\|g - S_j\|_{C[0,1]} \leq 2h^\alpha, \quad h := \frac{1}{k}. \tag{21}$$

In other words, the set K_α can be covered by 3^k balls in $C[0,1]$ of radius $2k^{-\alpha}$ with centers from $\mathcal{S}(\xi)$.

Proof We consider the following set $\mathcal{P}$ of patterns from $\mathcal{S}(\xi)$. For T to be in $\mathcal{P}$, we require that $T(\xi_j) = m_j h^\alpha$, with $m_0, \ldots, m_k$ integers satisfying the conditions

$$m_0 = m_k = 0, \quad |m_j - m_{j-1}| \leq 1, \quad j = 1, \ldots, k. \tag{22}$$

There are at most 3^k such patterns, i.e., $\#(\mathcal{P}) \leq 3^k$.

For the proof of our claim, given $g \in K_\alpha$, we first notice that $|g(\xi_j) - g(\xi_{j-1})| \leq h^\alpha$, $j = 1, \ldots, k$. We then approximate g by the CPwL function $S \in \mathcal{S}(\xi)$, where the values $S(\xi_j)$ are of the form $\beta_j h^\alpha$, $\beta_j \in \mathbb{Z}$, and are chosen so that $S(\xi_j) = \beta_j h^\alpha$ is the closest to $g(\xi_j)$, $j = 1, \ldots, k$. Note that this gives $\beta_0 = \beta_k = 0$ since $g(\xi_0) = 0 = g(\xi_k)$ and

$$\big|S(\xi_j) - g(\xi_j)\big| \leq h^\alpha/2. \tag{23}$$

When assigning the values $S(\xi_j)$, starting with $S(\xi_0) = 0$ and moving from left to right, if it happens that there are two possible choices for β_j (which happens if $g(\xi_j) \pm h^\alpha/2$ is an integer multiple of h^α), we select the β_j that is closest to the already determined β_{j-1}. Since

$$\begin{aligned} \big|\beta_j - \beta_{j-1}\big| h^\alpha &= \big|S(\xi_j) - S(\xi_{j-1})\big| \\ &\leq \big|S(\xi_j) - g(\xi_j)\big| + \big|g(\xi_j) - g(\xi_{j-1})\big| + \big|g(\xi_{j-1}) - S(\xi_{j-1})\big| \\ &\leq h^\alpha/2 + h^\alpha + h^\alpha/2 = 2h^\alpha, \end{aligned}$$

we have $|\beta_j - \beta_{j-1}| \leq 2$. But the case of equality is not possible since it would mean that at step j we have not selected β_j to be the closest to β_{j-1}. Therefore $|\beta_j - \beta_{j-1}| \leq 1$, and thus (22) holds, i.e., the constructed approximant S is a pattern from $\mathcal{P}$. Finally, we notice that any pattern from $\mathcal{P}$ has slopes with absolute value at most $h^{\alpha-1}$. Hence, for any $x \in [0,1]$, picking the point ξ_j the closest to x, we have

$$\begin{aligned} |g(x) - S(x)| &\leq |g(x) - g(\xi_j)| + |g(\xi_j) - S(\xi_j)| + |S(\xi_j) - S(x)| \\ &\leq (h/2)^\alpha + h^\alpha/2 + h^{\alpha-1}(h/2) \leq 2h^\alpha, \end{aligned}$$

where we used (23) and the fact that $|x - \xi_j| \le h/2$. Taking the maximum over $x \in [0, 1]$ establishes (21) and concludes the proof. □

The following theorem proves an estimate like (19) for Lip α spaces.

Theorem 7.3 *Let $W \ge 8$. If $X = C[0, 1]$ and $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, then*

$$\sigma_m(f, \overline{\Upsilon})_X \le C(W)\frac{|f|_{\mathrm{Lip}\,\alpha}}{(m \ln m)^\alpha}, \quad m \ge 2. \tag{24}$$

Proof Without loss of generality, we can assume that $|f|_{\mathrm{Lip}\,\alpha} = 1$. Fixing f and m, we first choose T as the piecewise linear function which interpolates f at the equally spaced points $x_0, \dots, x_m$, where $x_i := i/m$, $i = 0, \dots, m$.

Since f and T agree at the endpoints of the interval $J_i := [x_i, x_{i+1}]$, the slope of T on J_i has absolute value at most $m^{1-\alpha}$. Therefore,

$$|T(x) - T(y)| \le m^{1-\alpha}|x - y| \le |x - y|^\alpha, \quad x, y \in J_i,$$

and hence, T is also in Lip α with semi-norm at most one on each of these intervals.

We now define $g := f - T$ and write $g = \sum_{i=1}^m g\chi_{J_i}$. Each $g_i := g\chi_{J_i}$ is a function in Lip α with $|g_i|_{\mathrm{Lip}\,\alpha} \le 2$. Let k be the largest integer such that $3^k k \le m$ and let $\mathcal{P} = \{S_1, \dots, S_{3^k}\}$ be the set of the 3^k patterns given by Lemma 7.2. Applying this lemma to each of the functions $\bar{g}_i : [0, 1] \to \mathbb{R}$, defined by $\bar{g}_i(x) := 2^{-1}m^\alpha g_i((x+i)/m) \in K_\alpha$, we find a pattern $S_{j_i} \in \mathcal{P}$, $S_{j_i} : [0, 1] \to \mathbb{R}$, such that

$$\|\bar{g}_i - S_{j_i}\|_{C[0,1]} \le 2k^{-\alpha}.$$

Shifting back to the interval J_i provides a function $S_{j_i} \in \mathcal{P}$ such that

$$|g_i(x) - 2m^{-\alpha}S_{j_i}(m(x - x_i))| \le 4(km)^{-\alpha}, \quad x \in J_i,$$

and therefore the function $\hat{T}$ given by

$$\hat{T}(x) := T(x) + 2m^{-\alpha}\sum_{i=1}^m S_{j_i}(m(x - x_i))\chi_{J_i}(x) \tag{25}$$

approximates f to accuracy $4(km)^{-\alpha}$ in the uniform norm.

For each $j = 1, \dots, 3^k$, we consider the (possibly empty) set of indices $\Lambda_j = \{i \in \{1, \dots, m\} : j_i = j\}$. We have

$$\hat{T} = T + \sum_{j=1}^{3^k} T_j, \qquad \text{where} \quad T_j := 2m^{-\alpha}\sum_{i\in\Lambda_j} S_j(m(x - x_i)).$$

Since $T \in \Sigma_m$, Remark 3.3 says that T belongs to $\overline{\Upsilon}^{W,L_0}$ with either $W^2L_0 \asymp n(W, L_0) \le C'm$ or $L_0 = 2$. According to Theorem 5.1, each function T_j is in

$\overline{\Upsilon}^{W,L_j}$ with either $W^2 L_j \asymp n(W, L_j) \le C_1(k + m_j) + C_2 W^2$ or $L_j = 2$, where $m_j := |\Lambda_j|$. Therefore, in view of (14), we derive that $\hat{T}$ belongs to $\overline{\Upsilon}^{W,L}$ with $L = L_0 + \sum_{j=1}^{3^k} L_j$, and

$$L = L_0 + \sum_{j=1}^{3^k} L_j \le \frac{1}{W^2}\left(C'm + C_1 3^k k + C_1 \sum_{j=1}^{3^k} m_j\right) + C_3 3^k \le \left\lceil \frac{\tilde{C}_1}{W^2} + \tilde{C}_2 \right\rceil m = c(W)m,$$

where we have used the facts that $3^k k \le m$ and $\sum_{j=1}^{3^k} m_j = m$. This shows that $\hat{T} \in \overline{\Upsilon}_{c(W)m}$ and in turn that

$$\sigma_{c(W)m}(f, \overline{\Upsilon})_{C[0,1]} \le \|f - \hat{T}\|_{C[0,1]} \le \frac{4}{(km)^\alpha} \le \frac{\widetilde{C}}{(m \ln m)^\alpha},$$

where in the last inequality we have used that $k \ge c \ln m$ since $3^{k+1}(k+1) > m$. Up to the change of m in $c(W)m$, this is the result announced in (24). □

7.2.2 Other Classical Smoothness Spaces via K-Functionals

In this section, we want to show how the existing theorems on approximating classical smoothness classes with deep networks have a simple extension to more general smoothness classes using methods of K-functionals. Since we do not wish to delve too deeply into the theory of smoothness spaces in the present paper, we illustrate this with just one example.

Theorem 7.4 *Let $W \ge 12$. If $X = C[0,1]$ and $f \in C[0,1]$ satisfies $f' \in L_p[0,1]$, $1 \le p \le \infty$, then*

$$\sigma_m(f, \overline{\Upsilon})_X \le C(W)\|f'\|_{L_p} m^{-2+1/p}, \quad m \ge 2. \tag{26}$$

Proof When $p = \infty$, (26) follows from (20) since $f' \in L_\infty[0,1]$ is equivalent to $f \in \text{Lip } 1$ and $|f|_{\text{Lip } 1} = \|f'\|_{L_\infty}$. The case $p = 1$ follows from

$$\sigma_{\gamma m}(f, \overline{\Upsilon})_X \le \sigma_m(f, \Sigma)_X \le \|f'\|_{L_1} m^{-1}, \quad m \ge 1. \tag{27}$$

Here, the first inequality follows from Theorem 7.1 and the second inequality is a consequence of an estimate (already mentioned) for CPwL approximation of f with $f' \in L_1[0,1]$. Now, given $1 < p < \infty$ and $f \in C[0,1]$ with $f' \in L_p[0,1]$, for any $t > 0$, we can write

$$f = f_0 + f_1,$$

where

$$\max\{\|f_1'\|_{L_1}, t\|f_0'\|_{L_\infty}\} \le \|f_1'\|_{L_1} + t\|f_0'\|_{L_\infty} \le 2\|f'\|_{L_p} t^{1-1/p}.$$

This is a well-known and easily derived result for K-functionals. We take $t := m^{-1}$ and find

$$\begin{aligned}\sigma_{2\gamma m}(f, \overline{\Upsilon})_X &\le \sigma_{\gamma m}(f_0, \overline{\Upsilon})_X + \sigma_{\gamma m}(f_1, \overline{\Upsilon})_X \\ &\le C(W)\{\|f_0'\|_{L_\infty} m^{-2} + \|f_1'\|_{L_1} m^{-1}\} \\ &\le C(W)\|f'\|_{L_p}\{m^{-2}t^{-1/p} + m^{-1}t^{1-1/p}\} \\ &\le C(W)\|f'\|_{L_p} m^{-2+1/p},\end{aligned}$$

where we used the summation property for the elements of the family $\overline{\Upsilon}$. □

Remark 7.1 This theorem is not contained in the existing results mentioned above. One could use the existing results together with the Sobolev embedding which says that $f' \in L_p$ implies f is in Lip $(1-1/p)$ and obtain the rate $\mathcal{O}(m^{-2+2/p})$. However, this is worse than that given in the theorem. The theorem shows an improvement in the approximation rate when using deep networks in that the rate $\mathcal{O}(m^{-1})$ obtained when using Σ_m is now replaced by $m^{-2+1/p}$ when using deep networks. Note however that this improved rate lessens as we near the Sobolev embedding line.

7.3 The Power of Depth

In this subsection, we highlight several other classes of functions whose approximation rates by neural networks far exceed their approximation rates by free knots linear splines or any other standard approximation family. Our constructions are based on variants of the following simple observation.

Proposition 7.5 *For functions $f_k \in \overline{\Upsilon}_k$ satisfying $\|f_k\|_{C[0,1]} = 1$ for all $k \ge 1$ and for a sequence $(\beta_k)_{k\ge1}$ in $\ell_1(\mathbb{N})$, the function*

$$F := \sum_{k\ge1} \beta_k f_k$$

has approximation error satisfying

$$\sigma_{m^2}(F, \overline{\Upsilon})_{C[0,1]} \le \sum_{k>m} |\beta_k|, \quad m \ge 1.$$

Proof The function $S_m := \sum_{k=1}^m \beta_k f_k$ belongs to $\overline{\Upsilon}_{m^2}$, thanks to the summation and inclusion properties for $\overline{\Upsilon}$. A triangle inequality gives

$$\|F - S_m\|_{C[0,1]} \le \sum_{k>m} |\beta_k|,$$

and the statement follows immediately. □

Remark 7.2 When the functions f_k are related to one another, the proposition can be improved by replacing m^2 with a smaller quantity. For example, if $f_k = \phi^{\circ k}$ for a fixed function ϕ in $\Upsilon^{\mathrm{w},\ell}$, with width $2 \le \mathrm{w} \le W - 2$ and fixed depth ℓ, then Proposition 4.3 reveals that m^2 can be changed to ℓm.

We now present some classes of such functions F that are well-approximated by ReLU networks. For the most part, these functions cannot be well-approximated by standard approximation families.

7.3.1 The Takagi Class of Functions

For our first set of examples, let us recall that functions of the form

$$F = \sum_{k \ge 1} t^k g \circ \psi^{\circ k}, \quad |t| < 1, \tag{28}$$

with $\psi : [0,1] \to [0,1]$ and $g : [0,1] \to \mathbb{R}$, provide primary examples of self similar functions and dynamical systems [35]. If $g \in \Upsilon^{W_1,\ell}$ and $\psi \in \Upsilon^{W_2,\ell}$, with $W_1 + W_2 = W$, Proposition 4.4 implies that the partial sum $S_m := \sum_{k=1}^m t^k g \circ \psi^{\circ k}$ belongs to $\overline{\Upsilon}^{W+2,\ell(m+1)} = \overline{\Upsilon}_{\ell(m+1)}$. Therefore, in this case, the function F defined via (28) is approximated by the partial sum S_m with exponential accuracy by ReLU networks,

$$\sigma_{\ell(m+1)}(F, \overline{\Upsilon})_{C[0,1]} \le C \frac{|t|^{m+1}}{1 - |t|}, \quad m \ge 1.$$

Now, we consider a special class of functions. For this purpose, we recall that the hat function $H \in \Upsilon^{2,1}$ and its k-fold composition $H^{\circ k} := H \circ H \circ \cdots \circ H$, according to the composition property (13), belongs to $\overline{\Upsilon}^{2,k}$. On the other hand, the same function $H^{\circ k}$ is in Σ_n only if n is exponential in k. For an absolutely summable sequence $(c_k)_{k \ge 1}$ of real numbers, we consider continuous functions F of the form

$$F := \sum_{k \ge 1} c_k H^{\circ k},$$

approximations to which are produced by the special networks shown in Fig. 16. The collection of all such functions is called the Takagi class. It contains a number of interesting and important examples. A good source of information on the Takagi class is [1], from which the two examples below are taken.

For the first example, we take $c_k := 2^{-k}$, which gives the Takagi function

$$T := \sum_{k \ge 1} 2^{-k} H^{\circ k}.$$

From Remark 7.2, we have

$$\sigma_m(T, \overline{\Upsilon})_X \le 2^{-m}, \quad m \ge 1,$$

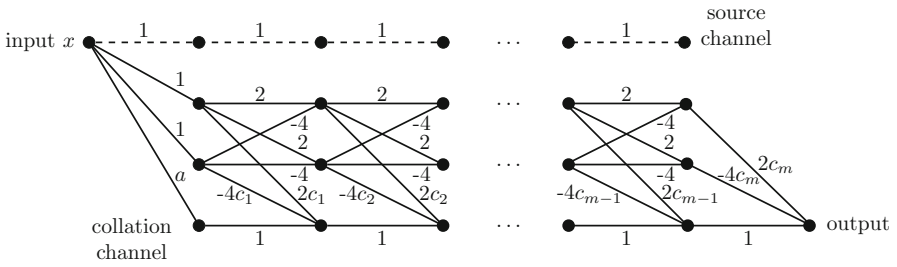

Fig. 16 Computation graph associated with the approximation of the Takagi class

and so theoretically T can be approximated with exponential accuracy by ReLU networks with roughly $W^2 m$ parameters. In practice, see Fig. 16, we can approximate it using m parameters. However, T is nowhere differentiable and so it has very little smoothness in the classical sense. This means that all of the traditional methods of approximation will fail miserably to approximate it. Note that the function T has self similarity, in that it satisfies a simple refinement equation.

Other examples take a highly lacunary sequence of coefficients and thereby construct functions in the Takagi class that do not satisfy a Lipschitz condition of any order and yet they can be approximated to exponential accuracy by $\overline{\Upsilon}$. Many functions from the Takagi class are fractals, in the sense that the Hausdorff dimension of their graph is strictly greater than one.

We do not go into the Takagi class more deeply but refer the reader to [1,14] where the properties and applications of the Takagi functions are given as well as numerous examples of similar constructions. The main point to draw from these examples is that the approximation classes $\mathcal{A}^r$ for r large contain many functions which are not smooth in any classical sense. This point was also made in [12], where the authors show that oscillatory textures can be approximated with exponential accuracy, see Proposition IX.2, and that see Proposition IX.3, the Weierstrass function

$$W_{p,a}(x) := \sum_{k=0}^{\infty} p^k \cos(a^k \pi x), \quad p \in (0, 1/2), \quad a \in \mathbb{R}^+, \quad ap \geq 1,$$

can be approximated by $S \in \Upsilon^{20, Cn^3}$ with exponential accuracy 2^{-n}, that is

$$\|W_{p,a} - S\|_{C[-1,1]} \leq 2^{-n}.$$

7.3.2 Analytic Functions

Another example in the Takagi class is the function, see [36],

$$x(1-x) = \sum_{k \geq 1} 4^{-k} H^{\circ k}. \tag{29}$$

This formula is used as a starting point to show that analytic functions are well-approximated by deep neural networks (see [11,21,36]), as we briefly discuss below.

It follows from (29) that the function x^2 is approximated with exponential accuracy by ReLU networks From this, one derives that all power functions x^k also are approximated with exponential accuracy. Then, using the summation property, one concludes that analytic functions and functions in Sobolev spaces are approximated with the same accuracy as their approximation by algebraic polynomials, up to logarithmic factors. Similarly, we can approximate functions on $[0, 1]$ from their power series representation. The point we emphasize here is the flexibility of ReLU networks. On one hand, classes of functions with little classical smoothness are well approximated by ReLU networks (some even with exponential accuracy), while on the other hand, ReLU networks still retain the accuracy of well-known approximation methods for classically smooth functions.

8 Neural Network Approximation as Manifold Approximation

Up to this point, we have reflected on the expressive power and the corresponding approximation power of deep ReLU networks. In other words, we wondered how well the best approximation from $\overline{\Upsilon}_\ell$ to a target function performs. An important practical issue is the construction of reasonable methods of approximation that yield near-best approximations to any given target function $f \in X$ with, e.g., $X = C[0, 1]$.

To discuss this problem, we need to formulate what would be considered a reasonable approximation procedure. The set $\overline{\Upsilon}_\ell$ is described by roughly $W^2\ell$ parameters, which are identified by a point in $\mathbb{R}^m$, $m = CW^2\ell$. We let $M = M_m$ be the mapping that sends $z \in \mathbb{R}^m$ to the function $M(z)$ generated by the neural network with the chosen parameters z. We view the collection $\mathcal{M} = \mathcal{M}_m$ of all $M(z)$, $z \in \mathbb{R}^m$, as an m-dimensional manifold. Here, in contrast to the usual use of the term manifold in topology, we do not assume a priori any particular smoothness of the mapping M. In this context, we also view any approximation method as providing a mapping $a = a_m : X \to \mathbb{R}^m$ which, for a given $f \in X$, selects the parameters of the network used to approximate f. The approximation to f is then

$$A_m(f) = M_m(a_m(f)), \quad m \geq 0.$$

A fundamental question for both theory and numerical practice is what conditions to impose on a_m and M_m so that the resulting scheme A_m is reasonable. In keeping with the notion of numerical stability, we could require that each of these mappings is a Lip 1 function with a fixed Lipschitz constant Γ independent of m. This means that there is a norm $\|\cdot\|$ on $\mathbb{R}^m$ (typically an ℓ_p norm) such that for any $f_1, f_2 \in X$,

$$\|a_m(f_1) - a_m(f_2)\| \leq \Gamma \|f_1 - f_2\|_X.$$

The stability of M_m means that, for any $z_1, z_2 \in \mathbb{R}^m$,

$$\|M_m(z_1) - M_m(z_2)\|_X \leq \Gamma \|z_1 - z_2\|.$$

One can lessen the demand on numerical stability to requiring only that the mappings a_m and M_m are continuous, not necessarily Lipschitz. This weaker assumption was used in the definition of manifold widths [8]. This manifold width of a compact set $K \subset X$ is defined as

$$\delta_m(K) := \inf_{(a,M)} \sup_{f\in K} \|f - M(a(f))\|_X,$$

where the infimum is taken over all continuous maps $a : K \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to X$. It is shown in [9] that this milder requirement still puts a restriction on how well sets characterized by classical smoothness can be approximated. For example, if K is the unit ball of Lip α, then $\delta_m(K) \geq Cm^{-\alpha}$. Therefore, the improvements discussed in Sect. 7.2 cannot be obtained with continuous selection of parameters. This lack of continuity for some approximation schemes was also recognized in [18]. This may be a crucial point in the framing of results on the instability of certain methods for constructing deep network approximations to target functions from data via optimization methods (such as least squares or constrained least squares methods).

9 Appendix

9.1 The Matrices of Lemma 3.3

In order to explicitly write the affine transforms $A^{(1)}$ and $A^{(2)}$ that determine the ReLU net, we describe here one of the possible ways to partition the set of indices Λ so that the constant sign and separation properties are satisfied. To do this, we first consider Λ_+ and only the main breakpoints ξ_j with indices j for which $j \mod 3 = \ell$. We collect into the set $\Lambda_i^{\ell,+}$ all indices $k \in \Lambda_+$ that correspond to the i-th hat function $H_{i,j}$ associated with a principal breakpoint ξ_j with the above mentioned property. Recall that there are q hat functions $H_{i,j}$ associated with each principal breakpoint ξ_j. We do this for every $\ell = 0, 1, 2$, and Λ_-, and we get the partition

$$\Lambda_i^{\ell,+} := \{s : s \in \Lambda_+ \text{ and } \phi_s = H_{i,j} \text{ with } j \mod 3 = \ell\},$$
$$\Lambda_i^{\ell,-} := \{s : s \in \Lambda_- \text{ and } \phi_s = H_{i,j} \text{ with } j \mod 3 = \ell\},$$

where $\ell = 0, 1, 2$, $i = 1, \ldots, q$. The matrices that determine the special network are

$$M^{(1)} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 0 \end{bmatrix}^\top, \quad b^{(1)} = \begin{bmatrix} 0 & \xi_1 & \ldots & \xi_{W-2} & 0 \end{bmatrix}^\top,$$
$$M^{(2)} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ m_{2,1}^{(2)} & m_{2,2}^{(2)} & \ldots & m_{2,W-1}^{(2)} & 0 \\ \ldots & \ldots & & & \\ m_{W-1,1}^{(2)} & m_{W-1,2}^{(2)} & \ldots & m_{W-1,W-1}^{(2)} & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, \quad b^{(2)} = \begin{bmatrix} 0 \\ b_2^{(2)} \\ \ldots \\ b_{W-2}^{(2)} \\ 0 \end{bmatrix}$$
$$M^{(3)} = \begin{bmatrix} 0 & \varepsilon_1^{(3)} & \ldots & \varepsilon_{W-2}^{(3)} & 1 \end{bmatrix}, \quad b^{(3)} = 0,$$

where $\varepsilon_k^{(3)} = 1$ if $\Lambda_k \subset \Lambda_+$, $\varepsilon_k^{(k)} = -1$ if $\Lambda_k \subset \Lambda_-$, and $\varepsilon_k^{(3)} = 0$ if $\Lambda_k = \emptyset$. Next, we demonstrate how to find the entries of one row in $M^{(2)}$. The rest of the rows are computed likewise. The index $k = 1, \dots, W-2$ in (10) corresponds to a different labeling of the index set

$$\{(i, \ell, +), (i, \ell, -),\ i = 1, \dots, q,\ \ell = 0, 1, 2\},$$

of the particular partition we work with here. We take the index $(1, 1, +)$ and compute the corresponding $\tilde{T}$,

$$\tilde{T} := T_{(1,1+)} = \sum_{s \in \Lambda_1^{1,+}} c_s \phi_s = [\tilde{S}]_+,$$

see Fig. 6, where $\tilde{S}$ is a CPwL function with breakpoints the principal breakpoints $\xi_1, \dots, \xi_{W-2}$, with the property

$$\tilde{S}(\xi_{3s+1}) = c_{3s+1}, \quad \tilde{S}(x_{(3s+1)q-1}) = \tilde{S}(x_{(3s+1)q+1}) = 0, \quad s = 0, \dots, \left\lfloor \frac{W-3}{3} \right\rfloor.$$

Then, the entries in the second row in $M^{(2)}$ and $b^{(2)}$ are the coefficients from the representation

$$\tilde{S}(x) = m_{2,1}^{(2)} x + \sum_{j=2}^{W-2} m_{2,j}^{(2)} (x - \xi_j)_+ + b_2^{(2)}.$$

9.2 Theorem 3.1, Case $4 \le W \le 7$

In this case, we have to show that for every $n \ge 1$ the set Σ_n of free knot linear splines with n breakpoints is contained in the set $\Upsilon^{W,L}$ of functions produced by width-W and depth-L ReLU networks where

$$L = \begin{cases} 2\left\lceil \frac{n}{2(W-2)} \right\rceil, & n \ge 2(W-2), \\ 2, & n < 2(W-2), \end{cases}$$

and whose number of parameters

$$n(W, L) \le \begin{cases} Cn, & n \ge 2(W-2), \\ W^2 + 4W + 1, & n < 2(W-2), \end{cases}$$

where C is an absolute constant. We start with the case $W - 2 = 2$. Given $n \ge 4$, we choose $L := \lceil \frac{n}{4} \rceil$. If $n < 4L$, we add artificial breakpoints so that we represent $T \in \Sigma_n \subset \Sigma_{4L}$ as

$$T(x) = ax + b + \sum_{j=1}^{4L} c_j (x - \xi_j)_+ = ax + b + \sum_{j=1}^{2L} S_j,$$
$$S_j := c_{2j-1}(x - \xi_{2j-1})_+ + c_{2j}(x - \xi_{2j})_+.$$

Now we can construct the special network with output $\overline{\underline{\Upsilon}}^{4,2L}$ that generates T via the successive transformations $A^{(j)}$ given by the matrices

$$M^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^\top, \quad b^{(1)} = \begin{bmatrix} 0 & -\xi_1 & -\xi_2 & 0 \end{bmatrix}^\top,$$

The jth layer, $j = 2, \ldots, 2L$, produces S_{j-1} in its CC node via the matrix

$$M^{(j)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & c_{2j-3} & c_{2j-2} & 1 \end{bmatrix}, \quad b^{(j)} = \begin{bmatrix} 0 \\ -\xi_{2j-1} \\ -\xi_{2j} \\ 0 \end{bmatrix}.$$

Finally, the output layer is given by the matrix

$$M^{(2L)} = \begin{bmatrix} a & c_{2L-1} & c_{2L} & 1 \end{bmatrix}, \quad b^{(2L)} = b,$$

where the first entry a and the bias b account for the linear function $ax + b$ in T. In this case, we have $\Sigma_{4L} \subset \overline{\underline{\Upsilon}}^{4,2L} \subset \Upsilon^{4,2L}$, with number of parameters

$$n(4, 2L) = 40L - 7 = 40 \left\lceil \frac{n}{4} \right\rceil - 7 < 10n + 33 < 19n, \quad n \geq 4.$$

For the case $n < 4$, we again add artificial breakpoints so that we represent $T \in \Sigma_n \subset \Sigma_4$ as

$$T(x) = ax + b + \sum_{j=1}^{4} c_j (x - \xi_j)_+ = ax + b + \sum_{j=1}^{2} S_j,$$
$$S_j := c_{2j-1}(x - \xi_{2j-1})_+ + c_{2j}(x - \xi_{2j})_+,$$

and as above construct a special network with output $\overline{\underline{\Upsilon}}^{4,2}$ for which $\Sigma_n \subset \overline{\underline{\Upsilon}}^{4,2}$, and whose parameters

$$n(4, 2) = 33 = W^2 + 4W + 1, \quad W = 4.$$

Now, for the case $(W - 2) \in \{3, 4, 5\}$, let us first consider $n \geq 2(W - 2)$ and take $L := \left\lceil \frac{n}{2(W-2)} \right\rceil$. If $n < 2(W - 2)L$, we add artificial breakpoints so that we represent $T \in \Sigma_n \subset \Sigma_{2(W-2)L}$. We do the same construction as in the case $W - 2 = 2$, by

dividing the indices $\{1, \ldots, 2(W-2)L\}$ into $2L$ groups of $W-2$ numbers, as shown in

$$T(x) = ax + b + \sum_{j=1}^{2(W-2)L} c_j (x - \xi_j)_+ = ax + b + \sum_{j=1}^{2L} S_j,$$
$$S_j := \sum_{i=0}^{W-3} c_{(W-2)j-i} (x - \xi_{(W-2)j-i})_+,$$

and execute the same construction as before by concatenating the networks producing S_j. In this case, we have $\Sigma_n \subset \Sigma_{2(W-2)L} \subset \overline{\Upsilon}^{W,2L}$, and when $n \geq 2(W-2)$,

$$\begin{aligned} n(W, 2L) &= 2W(W+1) \left\lceil \frac{n}{2(W-2)} \right\rceil - (W-1)^2 + 2 \\ &< \frac{W(W+1)}{W-2} n + W^2 + 4W + 1 < 25n. \end{aligned}$$

When $n < 2(W-2)$, we again add artificial breakpoints so that we represent $T \in \Sigma_n \subset \Sigma_{2(W-2)}$ as

$$T(x) = ax + b + \sum_{j=1}^{2(W-2)} c_j (x - \xi_j)_+ = ax + b + \sum_{j=1}^{2} S_j,$$
$$S_j := \sum_{i=0}^{W-3} c_{(W-2)j-i} (x - \xi_{(W-2)j-i})_+,$$

and as above generate a special network that outputs $\overline{\Upsilon}^{W,2}$ with depth $L = 2$ for which $\Sigma_n \subset \overline{\Upsilon}^{W,2}$, and whose parameters

$$n(W, 2) = 2W(W+1) - (W-1)^2 + 2 = W^2 + 4W + 1, \quad n < 2(W-2).$$

This completes the proof. □

9.3 Proof of Theorem 4.1

Proof Note that for every k-tuple $(\tilde{S}_k, \cdots, \tilde{S}_1) \in \Sigma_{n_k} \times \cdots \times \Sigma_{n_1}$, we can find another k-tuple $(S_k, \ldots, S_1) \in \Sigma_{n_k} \times \cdots \times \Sigma_{n_1}$, which we call a representative of the composition, with the properties:

- $S_j([0,1]) \subset [0,1]$, $j = 1, \ldots, k-1$.
- $\tilde{S}_k \circ \cdots \circ \tilde{S}_1 = S_k \circ \cdots \circ S_1$.

Indeed, if we denote by $m_1 := \min_{x\in[0,1]} \tilde{S}_1(x)$, $M_1 := \max_{x\in[0,1]} \tilde{S}_1(x)$, define inductively

$$m_j := \min_{x\in[m_{j-1},M_{j-1}]} \tilde{S}_j, \quad M_j := \max_{x\in[m_{j-1},M_{j-1}]} \tilde{S}_j, \quad j = 2,\dots,k-1,$$

and consider the functions

$$\begin{aligned} S_1 &:= \frac{\tilde{S}_1 - m_1}{M_1 - m_1} \in \Sigma_{n_1}, \\ S_j(x) &:= \frac{\tilde{S}_j(x(M_{j-1} - m_{j-1}) + m_{j-1}) - m_j}{M_j - m_j} \in \Sigma_{n_j}, \quad j = 2,\dots,k-1, \\ S_k(x) &:= \tilde{S}_k(x(M_{k-1} - m_{k-1}) + m_{k-1}). \end{aligned}$$

The k-tuple $(S_k,\dots,S_1)$ will be a representative of the composition $\tilde{S}_k \circ \dots \circ \tilde{S}_1$. So, in going further, we will always assume that we are dealing with representatives of all compositions we consider and with ReLU networks that output these representatives.

Relation (11) follows from Proposition 4.2 and Theorem 3.1. Indeed, if we fix an element in $\Sigma^{n_k\circ\cdots\circ n_1} := \{\tilde{S}_k \circ \cdots \circ \tilde{S}_1 : \tilde{S}_j \in \Sigma_{n_j}, j = 1,\dots,k\}$ and consider its representative $(S_k,\dots,S_1)$, each S_j in the composition $S_k \circ \cdots \circ S_1$ can be produced by a ReLU network with width W and depth

$$L_j = 2\left\lceil \frac{n_j}{\lfloor \frac{W-2}{6} \rfloor (W-2)} \right\rceil,$$

and therefore, part **(i)** of Proposition 4.2 ensures that $S_k \circ \cdots \circ S_1 \in \Upsilon^{W,\sum_{j=1}^k L_j}$. A similar estimate as in the proof of Theorem 3.1 yields

$$n(W,L) < 34 \sum_{j=1}^{k} n_j + 2k(W^2 + W),$$

as desired.

To establish (12), for each $i = 1,\dots,m$, let us denote by $\mathcal{N}_i$ the ReLU network from (11) with width $W-2$ that produces the composition $S_{i,\ell_i} \circ \cdots \circ S_{i,1}$ and has depth

$$L_i = L(n_{i,\ell_i},\dots,n_{i,1}) = 2\sum_{j=1}^{\ell_i} \left\lceil \frac{n_{i,j}}{\lfloor \frac{W-4}{6} \rfloor (W-4)} \right\rceil.$$

Then, Proposition 4.2, part **(ii)** gives

$$S = \sum_{i=1}^{m} a_i S_{i,\ell_i} \circ \cdots \circ S_{i,1} \in \overline{\Upsilon}^{W,L},$$

with

$$L = \sum_{i=1}^{m} L_i = 2 \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \left\lceil \frac{n_{i,j}}{\lfloor \frac{W-4}{6} \rfloor (W-4)} \right\rceil.$$

A similar estimate as in the proof of Theorem 3.1 yields

$$n(W, L) < 44 \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} n_{i,j} + 2W(W+1) \sum_{i=1}^{m} \ell_i.$$

As discussed in Remark 3.1, $\overline{\Upsilon}^{W,L}$ can always be viewed as a subset of $\Upsilon^{W,L}$, and the proof is completed. □

9.4 Proof of Proposition 6.1

Let us first start with the notation

$$\mathbf{1}_{\{i=j\}} := \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and isolate the following technical observation.

Lemma 9.1 *For any nonnegative sequence $u \in \ell_2(\mathbb{N})$,*

$$\sum_{\substack{k,\ell \geq 1 \\ k \neq \ell}} u_k u_\ell \sum_{m,n \geq 0} \frac{1}{(2m+1)^2} \frac{1}{(2n+1)^2} \mathbf{1}_{\{(2m+1)k = (2n+1)\ell\}} \leq \frac{\pi^4}{192} \|u\|_2^2. \tag{30}$$

Proof For each integer $m \geq 0$, let us introduce the sequence $u^{(2m+1)} \in \ell_2(\mathbb{N})$ defined by

$$u_j^{(2m+1)} = \begin{cases} u_{\frac{j}{2m+1}}, & \text{if } j \in (2m+1)\mathbb{N}, \\ 0, & \text{if } j \notin (2m+1)\mathbb{N}, \end{cases}$$

i.e., we consider a new sequence obtained from the original one by separating every two consecutive terms with $2m$ zeros, starting with $2m$ zeros. We easily see that

$$\langle u^{(2m+1)}, u^{(2n+1)} \rangle = \sum_{j \in \mathbb{N}} u_j^{(2m+1)} u_j^{(2n+1)} = \sum_{k,\ell \in \mathbb{N}} u_k u_\ell \mathbf{1}_{\{(2m+1)k = (2n+1)\ell\}},$$

and in particular $\|u^{(2m+1)}\|_2^2 = \|u\|_2^2$ for every $m \geq 0$. Thus, the left-hand side of (30), which we denote by Σ, can be written as

$$\begin{aligned}\Sigma &= \sum_{\substack{m,n\geq 0\\ m\neq n}} \frac{1}{(2m+1)^2}\frac{1}{(2n+1)^2} \sum_{k,\ell\geq 1} u_k u_\ell \mathbf{1}_{\{(2m+1)k=(2n+1)\ell\}} \\ &= \sum_{\substack{m,n\geq 0\\ m\neq n}} \frac{1}{(2m+1)^2}\frac{1}{(2n+1)^2} \langle u^{(2m+1)}, u^{(2n+1)}\rangle \\ &= \Big\| \sum_{m\geq 0} \frac{1}{(2m+1)^2} u^{(2m+1)} \Big\|_2^2 - \sum_{m\geq 0} \frac{1}{(2m+1)^4}\|u\|_2^2. \end{aligned} \tag{31}$$

By a simple triangle inequality, we have, see [32], Chapter 2, Exercise 6,

$$\Big\| \sum_{m\geq 0} \frac{1}{(2m+1)^2} u^{(2m+1)} \Big\|_2 \leq \sum_{m\geq 0} \frac{1}{(2m+1)^2}\|u\|_2 = \frac{\pi^2}{8}\|u\|_2. \tag{32}$$

Moreover, it is well-known that see [32], Chapter 3, Exercise 8(a),

$$\sum_{m\geq 0} \frac{1}{(2m+1)^4} = \frac{\pi^4}{96}. \tag{33}$$

Substituting (32) and (33) into (31) yields the announced result. □

Proof of Proposition 6.1 We equivalently prove the result for the L_2-normalized version of the system $(\mathcal{C}_k, \mathcal{S}_k)_{k\geq 1}$, i.e., for $(\widetilde{\mathcal{C}}_k := \sqrt{3}\,\mathcal{C}_k, \widetilde{\mathcal{S}}_k := \sqrt{3}\,\mathcal{S}_k)_{k\geq 1}$. Let $(c_k, s_k)_{k\geq 1}$ denote the orthonormal basis for $L_2^0[0,1]$ made of the usual trigonometric functions

$$c_k(x) = \sqrt{2}\cos(2\pi kx), \qquad s_k(x) = \sqrt{2}\sin(2\pi kx), \qquad x \in [0,1].$$

It is routine to verify (by computing Fourier series) that

$$\mathcal{C} = \lambda \sum_{m\geq 0} \frac{1}{(2m+1)^2} c_{2m+1}, \qquad \mathcal{S} = \lambda \sum_{m\geq 0} \frac{(-1)^m}{(2m+1)^2} s_{2m+1},$$

for some constant $\lambda > 0$, from which one immediately obtains that, for any $k \geq 1$,

$$\widetilde{\mathcal{C}}_k = \mu \sum_{m\geq 0} \frac{1}{(2m+1)^2} c_{(2m+1)k}, \qquad \widetilde{\mathcal{S}}_k = \mu \sum_{m\geq 0} \frac{(-1)^m}{(2m+1)^2} s_{(2m+1)k},$$

for some constant $\mu > 0$. Notice that this implies $\widetilde{\mathcal{C}}_k \perp s_\ell$, $\widetilde{\mathcal{S}}_k \perp c_\ell$, and $\widetilde{\mathcal{C}}_k \perp \widetilde{\mathcal{S}}_\ell$ for all $k, \ell \geq 1$. Moreover, the normalization $\|\widetilde{\mathcal{C}}_k\|_{L_2[0,1]} = \|\widetilde{\mathcal{S}}_k\|_{L_2[0,1]} = 1$ gives

$$\mu^2 \sum_{m\geq 0} \frac{1}{(2m+1)^4} = 1, \qquad \text{i.e.,} \qquad \mu^2 \frac{\pi^4}{96} = 1.$$

Let us introduce operators $T_{\mathcal{C}}, T_{\mathcal{S}}$ defined for $v \in \ell_2(\mathbb{N})$ and $j \in \mathbb{N}$, by

$$T_{\mathcal{C}}(v)_j = \sum_{k\geq 1} v_k \langle \widetilde{\mathcal{C}}_k, c_j \rangle = \mu \sum_{k\geq 1} v_k \sum_{m\geq 0} \frac{1}{(2m+1)^2} \mathbf{1}_{\{(2m+1)k=j\}},$$
$$T_{\mathcal{S}}(v)_j = \sum_{k\geq 1} v_k \langle \widetilde{\mathcal{S}}_k, s_j \rangle = \mu \sum_{k\geq 1} v_k \sum_{m\geq 0} \frac{(-1)^m}{(2m+1)^2} \mathbf{1}_{\{(2m+1)k=j\}},$$

and let us first verify that these are well-defined operators from $\ell_2(\mathbb{N})$ to $\ell_2(\mathbb{N})$, i.e., that both $\|T_{\mathcal{C}} v\|_2$ and $\|T_{\mathcal{S}} v\|_2$ are finite when $v \in \ell_2(\mathbb{N})$. To do so, we observe that

$$\begin{aligned}\|T_{\mathcal{C}} v\|_2^2 &= \mu^2 \sum_{j\geq 1} \sum_{k,\ell\geq 1} v_k v_\ell \sum_{m,n\geq 0} \frac{1}{(2m+1)^2} \frac{1}{(2n+1)^2} \mathbf{1}_{\{(2m+1)k=j\}} \mathbf{1}_{\{(2n+1)\ell=j\}} \\ &= \Sigma_{(=)} + \Sigma_{(\neq)},\end{aligned}$$

where $\Sigma_{(=)}$ represents the contribution to the sum when k and ℓ are equal and $\Sigma_{(\neq)}$ represents the contribution to the sum when k and ℓ are distinct. We notice that

$$\Sigma_{(=)} = \sum_{k\geq 1} v_k^2 \, \mu^2 \sum_{m\geq 0} \frac{1}{(2m+1)^4} \sum_{j\geq 1} \mathbf{1}_{\{(2m+1)k=j\}} = \sum_{k\geq 1} v_k^2 \, \mu^2 \sum_{m\geq 0} \frac{1}{(2m+1)^4} = \sum_{k\geq 1} v_k^2.$$

Therefore, relying on Lemma 9.1, we obtain

$$\begin{aligned}&\big|\|T_{\mathcal{C}} v\|_2^2 - \|v\|_2^2\big| \\ &= \big|\Sigma_{(\neq)}\big| \leq \mu^2 \sum_{\substack{k,\ell\geq 1 \\ k\neq\ell}} |v_k||v_\ell| \sum_{m,n\geq 0} \frac{1}{(2m+1)^2} \frac{1}{(2n+1)^2} \sum_{j\geq 1} \mathbf{1}_{\{(2m+1)k=j\}} \mathbf{1}_{\{(2n+1)\ell=j\}} \\ &= \mu^2 \sum_{\substack{k,\ell\geq 1 \\ k\neq\ell}} |v_k||v_\ell| \sum_{m,n\geq 0} \frac{1}{(2m+1)^2} \frac{1}{(2n+1)^2} \mathbf{1}_{\{(2m+1)k=(2n+1)\ell\}} \\ &\leq \mu^2 \frac{\pi^4}{192} \|v\|_2^2 = \frac{1}{2}\|v\|_2^2. \end{aligned} \tag{34}$$

This clearly justifies that $\|T_{\mathcal{C}} v\|_2^2 < \infty$, and $\|T_{\mathcal{S}} v\|_2^2 < \infty$ is verified in a similar fashion. In fact, the inequality (34) and the analogous one for $T_{\mathcal{S}}$ show that

$$\|T_{\mathcal{C}}^* T_{\mathcal{C}} - I\|_{2\to 2} = \max_{\|v\|_2=1} |\langle v, (T_{\mathcal{C}}^* T_{\mathcal{C}} - I)v\rangle| \leq \frac{1}{2}, \qquad \|T_{\mathcal{S}}^* T_{\mathcal{S}} - I\|_{2\to 2} \leq \frac{1}{2}. \tag{35}$$

This ensures that the operators $T_{\mathcal{C}}^* T_{\mathcal{C}}$ and $T_{\mathcal{S}}^* T_{\mathcal{S}}$ are invertible. Let us assume for a while that the operators $T_{\mathcal{C}} T_{\mathcal{C}}^*$ and $T_{\mathcal{S}} T_{\mathcal{S}}^*$ are also invertible. Then we derive that $T_{\mathcal{C}}$ is invertible with inverse $(T_{\mathcal{C}}^* T_{\mathcal{C}})^{-1} T_{\mathcal{C}}^*$, since $(T_{\mathcal{C}}^* T_{\mathcal{C}})^{-1} T_{\mathcal{C}}^* T_{\mathcal{C}} = I$ is obvious and $T_{\mathcal{C}} (T_{\mathcal{C}}^* T_{\mathcal{C}})^{-1} T_{\mathcal{C}}^* = I$ is equivalent, by the invertibility of $T_{\mathcal{C}} T_{\mathcal{C}}^*$, to $T_{\mathcal{C}} T_{\mathcal{C}}^* T_{\mathcal{C}} (T_{\mathcal{C}}^* T_{\mathcal{C}})^{-1} T_{\mathcal{C}}^* = T_{\mathcal{C}} T_{\mathcal{C}}^*$, which is obvious. We derive that $T_{\mathcal{S}}$ is invertible in a similar fashion. From here, we can show that the system $(\widetilde{\mathcal{C}}_k, \widetilde{\mathcal{S}}_k)_{k \geq 1}$ spans $L_2^0[0, 1]$. Indeed, we claim that any $f \in L_2^0[0, 1]$ can be written, with $\alpha := (\langle f, c_j \rangle)_{j \geq 1}$ and $\beta := (\langle f, s_j \rangle)_{j \geq 1}$, as

$$f = \sum_{k \geq 1} (T_{\mathcal{C}}^{-1} \alpha)_k \widetilde{\mathcal{C}}_k + \sum_{k \geq 1} (T_{\mathcal{S}}^{-1} \beta)_k \widetilde{\mathcal{S}}_k.$$

This identity is verified by taking the inner product of partial sums with the c_j and s_j. Indeed,

$$\begin{aligned} \left\langle f - \sum_{k=1}^{K} (T_{\mathcal{C}}^{-1} \alpha)_k \widetilde{\mathcal{C}}_k - \sum_{k=1}^{K} (T_{\mathcal{S}}^{-1} \beta)_k \widetilde{\mathcal{S}}_k, c_j \right\rangle &= \langle f, c_j \rangle - \sum_{k=1}^{K} (T_{\mathcal{C}}^{-1} \alpha)_k \langle \widetilde{\mathcal{C}}_k, c_j \rangle - 0 \\ &= \left(T_{\mathcal{C}} (T_{\mathcal{C}}^{-1} \alpha) \right)_j - \left(T_{\mathcal{C}} (T_{\mathcal{C}}^{-1} \alpha)_{\{1, \ldots, K\}} \right)_j \\ &= \left(T_{\mathcal{C}} (T_{\mathcal{C}}^{-1} \alpha)_{\{K+1, \ldots\}} \right)_j. \end{aligned}$$

After a similar calculation with s_j, and in view of $\|T_{\mathcal{C}}\|_{2 \to 2}^2 = \|T_{\mathcal{C}}^* T_{\mathcal{C}}\|_{2 \to 2} \leq 3/2$, it follows that

$$\begin{aligned} &\left\| f - \sum_{k=1}^{K} (T_{\mathcal{C}}^{-1} \alpha)_k \widetilde{\mathcal{C}}_k - \sum_{k=1}^{K} (T_{\mathcal{S}}^{-1} \beta)_k \widetilde{\mathcal{S}}_k \right\|_{L_2[0,1]}^2 \\ &\leq \left\| T_{\mathcal{C}} (T_{\mathcal{C}}^{-1} \alpha)_{\{K+1, \ldots\}} \right\|_2^2 + \left\| T_{\mathcal{S}} (T_{\mathcal{S}}^{-1} \alpha)_{\{K+1, \ldots\}} \right\|_2^2 \\ &= \frac{3}{2} \left(\left\| (T_{\mathcal{C}}^{-1} \alpha)_{\{K+1, \ldots\}} \right\|_2^2 + \left\| (T_{\mathcal{S}}^{-1} \alpha)_{\{K+1, \ldots\}} \right\|_2^2 \right) \\ &\underset{K \to \infty}{\longrightarrow} 0, \end{aligned}$$

which confirms our claim. As for a normalized version of (17), it follows from (35) by noticing that

$$\begin{aligned} \left\| \sum_{k \geq 1} (a_k \widetilde{\mathcal{C}}_k + b_k \widetilde{\mathcal{S}}_k) \right\|_{L_2[0,1]}^2 - (\|a\|_2^2 + \|b\|_2^2) &= \left\| \sum_{k \geq 1} a_k \widetilde{\mathcal{C}}_k \right\|_{L_2[0,1]}^2 - \|a\|_2^2 \\ &\quad + \left\| \sum_{k \geq 1} b_k \widetilde{\mathcal{S}}_k \right\|_{L_2[0,1]}^2 - \|b\|_2^2, \end{aligned}$$

combined with the observation that

$$\left|\left\|\sum_{k\geq 1} a_k \widetilde{\mathcal{C}}_k\right\|^2_{L_2[0,1]} - \|a\|_2^2\right| = \left|\sum_{j\geq 1}\left(\sum_{k\geq 1} a_k \langle \widetilde{\mathcal{C}}_k, c_j\rangle\right)^2 - \|a\|_2^2\right| = \left|\sum_{j\geq 1}(T_{\mathcal{C}}a)_j^2 - \|a\|_2^2\right|$$
$$= \left|\|T_{\mathcal{C}}a\|_2^2 - \|a\|_2^2\right| = \left|\langle (T_{\mathcal{C}}^* T_{\mathcal{C}} - I)a, a\rangle\right| \leq \frac{1}{2}\|a\|_2^2,$$

and the similar observation that

$$\left|\left\|\sum_{k\geq 1} b_k \widetilde{\mathcal{S}}_k\right\|^2_{L_2[0,1]} - \|b\|_2^2\right| \leq \frac{1}{2}\|b\|_2^2.$$

We deduce that a normalized version of (17) holds with constants $\widetilde{c} = 1/2$ and $\widetilde{C} = 3/2$, hence (17) holds with $c = 1/6$ and $C = 1/2$.

It now remains to establish that the operators $T_{\mathcal{C}}T_{\mathcal{C}}^*$ and $T_{\mathcal{S}}T_{\mathcal{S}}^*$ are invertible, which we do by showing that

$$\|T_{\mathcal{C}}T_{\mathcal{C}}^* - I\|_{2\to 2} \leq \rho \qquad \text{and} \qquad \|T_{\mathcal{S}}T_{\mathcal{S}}^* - I\|_{2\to 2} \leq \rho \tag{36}$$

for some constant $\rho < 1$. We concentrate on the case of $T_{\mathcal{C}}$, as the case of $T_{\mathcal{S}}$ is handled similarly. We first remark that the adjoint of $T_{\mathcal{C}}$ is given, for any $v \in \ell_2(\mathbb{N})$ and $j \in \mathbb{N}$, by

$$T_{\mathcal{C}}^*(v)_j = \sum_{k\geq 1} v_k \langle \widetilde{\mathcal{C}}_j, c_k\rangle = \mu \sum_{k\geq 1} v_k \sum_{m\geq 0} \frac{1}{(2m+1)^2}\mathbf{1}_{\{(2m+1)j=k\}}.$$

We then compute

$$\|T_{\mathcal{C}}^* v\|_2^2 = \mu^2 \sum_{j\geq 1}\sum_{k,\ell\geq 1} v_k v_\ell \sum_{m,n\geq 0} \frac{1}{(2m+1)^2}\frac{1}{(2n+1)^2}\mathbf{1}_{\{(2m+1)j=k\}}\mathbf{1}_{\{(2n+1)j=\ell\}}$$
$$= \Sigma_{(=)}^* + \Sigma_{(\neq)}^*,$$

where $\Sigma_{(=)}^*$ represents the contribution to the sum when k and ℓ are equal and $\Sigma_{(\neq)}^*$ represents the contribution to the sum when k and ℓ are distinct. We notice that

$$\Sigma_{(=)}^* = \sum_{k\geq 1} v_k^2\, \mu^2 \sum_{m\geq 0} \frac{1}{(2m+1)^4} \sum_{j\geq 1} \mathbf{1}_{\{(2m+1)j=k\}}$$

satisfies, on the one hand,

$$\Sigma_{(=)}^* \leq \sum_{k\geq 1} v_k^2\, \mu^2 \sum_{m\geq 0} \frac{1}{(2m+1)^4} = \sum_{k\geq 1} v_k^2 = \|v\|_2^2,$$

and on the other hand, by considering only the summand for $m = 0$ and $j = k$,

$$\Sigma^*_{(=)} \geq \sum_{k\geq 1} v_k^2\, \mu^2 = \mu^2 \|v\|_2^2.$$

Moreover, we have

$$\begin{aligned}
\left|\Sigma^*_{(\neq)}\right| &\leq \mu^2 \sum_{\substack{k,\ell\geq 1\\ k\neq\ell}} |v_k||v_\ell| \sum_{m,n\geq 0} \frac{1}{(2m+1)^2}\frac{1}{(2n+1)^2} \sum_{j\geq 1} \mathbf{1}_{\{(2m+1)j=k\}}\mathbf{1}_{\{(2n+1)j=\ell\}} \\
&\leq \mu^2 \sum_{\substack{k,\ell\geq 1\\ k\neq\ell}} |v_k||v_\ell| \sum_{m,n\geq 0} \frac{1}{(2m+1)^2}\frac{1}{(2n+1)^2} \mathbf{1}_{\{(2m+1)\ell=(2n+1)k\}} \\
&\leq \mu^2 \frac{\pi^4}{192}\|v\|_2^2 = \frac{1}{2}\|v\|_2^2, \qquad (37)
\end{aligned}$$

where the last inequality used Lemma 9.1 again. Therefore, we obtain

$$\left|\langle (T_{\mathcal{C}} T^*_{\mathcal{C}} - I)v, v\rangle\right| = \left|\|T^*_{\mathcal{C}} v\|_2^2 - \|v\|_2^2\right| = \left|(\Sigma^*_{(=)} - \|v\|_2^2) + \Sigma^*_{(\neq)}\right| \leq (1-\mu^2)\|v\|_2^2 + \frac{1}{2}\|v\|_2^2.$$

Taking the maximum over all $v \in \ell_2(\mathbb{N})$ with $\|v\|_2 = 1$, we arrive at the result announced in (36) with $\rho := 1 - \mu^2 + 1/2 \leq 0.5145$. The proof is now complete. □

References

1. Allaart, P., Kawamura, K.: The Takagi function: a survey. Real Anal. Exchange **37**(1), 1–54 (2011-2012)
2. Bölcskei, H., Grohs, P., Kutyniok, G., Petersen, P.: Optimal approximation with sparsely connected deep neural networks. SIAM J. Math. Data Sci. **1**(1), 8–45 (2019)
3. Bronstein, M., Bruna, J., LeCun, Y., Szlam, A., Vandergheyn, P.: Geometric deep learning: going beyond Euclidean data. IEEE Signal Process. Mag. **34**(4), 18–42 (2017)
4. Chui, C., Li, X., Mhaskar, H.: Neural networks for localized approximation. Math. Comput. **63**, 607–623 (1994)
5. Cybenko, G.: Approximation by superpositions of a sigmoidal function. Math. Control Signals Syst. (MCSS) **2**(4), 303–314 (1989)
6. Daniely, A.: Depth separation for neural networks. Proc. Mach. Learn. Res. (COLT) **65**, 690–696 (2017)
7. DeVore, R.: Nonlinear approximation. Acta Numer. **7**, 51–150 (1998)
8. DeVore, R., Howard, R., Micchelli, C.: Optimal non-linear approximation. Manuscripta Math. **63**, 469–478 (1989)
9. DeVore, R., Kyriazis, G., Leviatan, D., Tikhomirov, V.M.: Wavelet compression and nonlinear n-widths. Adv. Comput. Math. **1**, 197–214 (1993)
10. DeVore, R., Lorentz, G.: Constructive Approximation. Springer, Berlin (1993)
11. Wang, E., Wang, Q.: Exponential convergence of the deep neural network approximation for analytic functions. Sci. China Math. **61**, 1733–1740 (2018)
12. Grohs, P., Perekerstenko, G., Ebrächter, D., Blöleskei, H.: Deep neural network approximation theory. IEEE Tran. Inf. Theory (2019). arXiv: 1901.02220
13. Hanin, B., Sellke, M.: Approximating continuous functions by ReLU nets of minimal width (2017). arXiv:1710.11278

14. Hata, M.: Fractals in Mathematics, Patterns and Waves-Qualitative Analysis of Nonlinear Differential Equations, pp. 259–278. Elsevier, Amsterdam (1986)
15. Hebb, D.: The Organization of Behavior: A Neuropsychological Theory. Wiley, Hoboken (1949)
16. Hornik, K., Stinchcombe, M., White, H.: Multilayer feedforward networks are universal approximators. Neural Netw. **2**(5), 359–366 (1989)
17. Lu, J., Shen, Z., Yang, H., Zhang, S.: Deep network approximation for smooth functions (2020). arXiv:2001.03040
18. Kainen, P., Kurkova, V., Vogt, A.: Approximation by neural networks is not continuous. Neurocomputing **29**, 47–56 (1999)
19. Krizhevsky, A., Sutskever, I., Hinton, G.: Imagenet classification with deep convolutional neural networks. NIPS (2012)
20. LeCun, Y., Bengio, Y., Hinto, G.: Deep learning. Nature **521**(7553), 436 (2015)
21. Liang, S., Srikant, R.: Why deep neural networks for function approximation? (2016). arXiv:1610.04161
22. Mehrabi, M., Tchamkerten, A., Yousefi, M.: Bounds on the approximation power of feedforward neural networks. ICML (2018)
23. Mhaskar, H.N., Poggio, T.: Deep vs. shallow networks: an approximation theory perspective. Anal. Appl. **14**, 829–848 (2016)
24. Poggio, T., Mhaskar, H., Rosasco, L., Miranda, B., Liao, Q.: Why and when can deep-but dot shallow-networks avoid the curse of dimensionality: a review. Int. J. Autom. Comput. **14**(5), 503–519 (2017)
25. Rosenblatt, F.: The perceptron: a probabilistic model for information storage and organization in the brain. Psychol. Rev. **65**(6), 386 (1958)
26. Safran, I., Shamir, O.: Depth-width tradeoffs in approximating natural functions with neural networks (2017). International Conference on Machine Learning. PMLR, 2017
27. Schwab, C., Zech, J.: Deep learning in high dimension. www.sam.math.ethz.ch/sam-reports/reports-final/reports2017/2017-57-rev2.pdf
28. Shaham, U., Cloninger, A., Coifman, R.: Provable approximation properties for deep neural networks. Appl. Comput. Harmonic Anal. **44**(3), 537–557 (2018)
29. Shen, Z., Yang, H., Zhang, S.: Deep network approximation characterized by number of neurons (2019). arXiv:1906.05497
30. Silver, D., Huang, A., Maddison, C., Guez, A., Sifre, L., Van Den Driessche, G., Schrittwieser, J., Antonoglou, I., Panneershelvam, V., Lanctot, M., et al.: Mastering the game of Go with deep neural networks and tree search. Nature **529**(7587), 484–489 (2016)
31. Silver, D., Schrittwieser, J., Simonyan, K., Antonoglou, I., Huang, A., Guez, A., Hubert, T., Baker, L., Lai, M., Bolton, A., et al.: Mastering the game of Go without human knowledge. Nature **550**(7676), 354 (2017)
32. Stein, E., Shakarchi, R.: Fourier Analysis. Princeton Lectures in Analysis, vol. 1. Princeton University Press, Princeton (2003)
33. Telgarsky, M.: Representation benefits of deep feedforward networks (2015). arXiv:1509.08101
34. Wu, Y., Schuster, M., Chen, Z., Le, Q., Norouzi, M., Macherey, W., Krikun, M., Cao, Y., Gao, Q., Macherey, K. et al.: Google's neural machine translation system: Bridging the gap between human and machine translatio (2016). arXiv:1609.08144
35. Yamaguti, M., Hata, M.: Weirstrass's function and Chaos. Hokkaido. Math. J. **12**, 333–342 (1983)
36. Yarotsky, D.: Error bounds for approximations with deep ReLU networks. Neural Netw. **94**, 103–114 (2017)
37. Yarotsky, D.: Quantified advantage of discontinuous weight selection in approximations with deep neural networks (2017). arXiv:1705.01365
38. Yarotsky, D.: Optimal approximation of continuous functions by very deep ReLU networks, In: Bubeck S, Perchet V, Rigollet P (eds.) Proceedings of the 31st Conference On Learning Theory, volume 75 of Proceedings of Machine Learning Resaerch, pp. 639–649. PMLR, 06–09 Jul 2018. arXiv:1802.03620

Authors and Affiliations

I. Daubechies[1] · R. DeVore[2] · S. Foucart[2] · B. Hanin[2,3,4] · G. Petrova[2]

✉ R. DeVore
rdevore@math.tamu.edu

I. Daubechies
ingrid@math.duke.edu

S. Foucart
foucart@math.tamu.edu

B. Hanin
bhanin@math.tamu.edu; bhanin@fb.com; bhanin@princeton.edu

G. Petrova
gpetrova@math.tamu.edu

1 Dept. of Mathematics, Duke University, Durham, NC 27708, USA

2 Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

3 Facebook AI Research, New York City, USA

4 Department of Operations Research and Financial Engineering, Princeton University, Sherrerd Hall, Charlton Street Princeton, NJ 08544, USA