# The necessity of depth for artificial neural networks to approximate certain classes of smooth and bounded functions without the curse of dimensionality

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#### Abstract

In this article we study high-dimensional approximation capacities of shallow and deep artificial neural networks (ANNs) with the rectified linear unit (ReLU) activation. In particular, it is a key contribution of this work to reveal that for all  $a, b \in \mathbb{R}$  with  $b - a \ge 7$  we have that the functions  $[a,b]^d \ni x = (x_1,\ldots,x_d) \mapsto \prod_{i=1}^d x_i \in \mathbb{R}$  for  $d \in \mathbb{N} = \{1,2,3,\ldots\}$ as well as the functions  $[a,b]^d \ni x = (x_1,\ldots,x_d) \mapsto \sin(\prod_{i=1}^d x_i) \in \mathbb{R}$  for  $d \in \mathbb{N}$  can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs with ReLU activation but can be approximated without the curse of dimensionality by sufficiently deep ANNs with ReLU activation. More specifically, we prove that in the case of shallow ReLU ANNs or deep ReLU ANNs with a fixed number of hidden layers and with the size of scalar real parameters of the approximating ANNs growing at most polynomially in the dimension  $d \in \mathbb{N}$  we have that the number of ANN parameters must grow at least exponentially in the dimension  $d \in \mathbb{N}$  while in the case of deep ReLU ANNs with the number of hidden layers growing in the dimension  $d \in \mathbb{N}$ we have that the number of scalar real parameters of the approximating ANNs grows at most polynomially in both the inverse of the prescribed approximation accuracy  $\varepsilon > 0$ and the dimension  $d \in \mathbb{N}$ , even if the absolute values of the ANN parameters are assumed to be uniformly bounded by one. We thus show that the product functions and the sine of the product functions are *polynomially tractable* approximation problems among the approximating class of deep ReLU ANNs with the number of hidden layers being allowed to grow in the dimension  $d \in \mathbb{N}$ . We establish the above outlined statement not only for the product functions and the sine of the product functions but also for other classes of target functions, in particular, for classes of uniformly globally bounded  $C^{\infty}$ -functions with compact support on any  $[a,b]^d$  with  $a \in \mathbb{R}, b \in (a,\infty)$ . Roughly speaking, in this work we lay open that simple approximation problems such as approximating the sine or cosine of products cannot be solved in standard implementation frameworks by shallow or insufficiently deep ANNs with ReLU activation in polynomial time, but can be approximated by sufficiently deep ReLU ANNs with the number of parameters growing at most polynomially.

# Contents

1	Introduction	4
2	Artificial neural network (ANN) calculus2.1Set of ANNs2.2Realization functions of ANNs2.3Parallelizations of ANNs2.4Identity ANNs2.5Compositions of ANNs2.6Sizes of parameters of ANNs	<b>11</b> 11 12 12 13 13 14
3	<ul> <li>Lower bounds for the minimal number of ANN parameters in the approximation of certain high-dimensional functions</li> <li>3.1 Lower bounds for approximations of product functions</li></ul>	17           17           20           22           28           30
4	<ul> <li>Upper bounds for the minimal number of ANN parameters in the approximation of certain high-dimensional functions</li> <li>4.1 Trade-off between the number and the size of ANN parameters</li></ul>	32            33            38            40            55           ons         73            79
5	<ul> <li>Lower and upper bounds for the minimal number of ANN parameters in a approximation of certain high-dimensional functions</li> <li>5.1 ANN approximations regarding high-dimensional product functions</li></ul>	the 86 87 89 95 ties 98

## 1 Introduction

Artificial neural network (ANN) approximations are ubiquitous in our digital world and appear in diverse areas, whether language processing (cf., e.g., Devlin et al. [9]), image classification (cf., e.g., Chen et al. [5]), predictive models for cancer diagnosis (cf., e.g., Sidey-Gibbons & Sidey-Gibbons [29]), risk assessment (cf., e.g., Paltrinieri et al. [22]), or biomedical imaging and signal processing (cf., e.g., Min et al. [19]) In such learning problems, ANNs are employed to approximate the target function which, roughly speaking, describes the best relationship of the input data to the output data in the considered learning problem. There are a large number of numerical simulation results which indicate that ANNs are comparatively well suited to approximate the target functions in such learning problems. The success of ANN approximations becomes even more remarkable if one takes into account that the target functions in the above named learning problems are usually extremely high-dimensional functions.

For example, in an object recognition problem, the 10000-dimensional unit cube  $[0, 1]^{10000}$ can be employed to represent the set of all grey-scale images with 100 × 100 pixels and the target function  $f: [0, 1]^{10000} \rightarrow [0, 1]$  of the considered learning problem is then a function from the 10000-dimensional unit cube  $[0, 1]^{10000}$  to the interval [0, 1] modelling for every image the probability that it contains a certain object, say, a car. Losely speaking, it is impossible to approximate such target functions by classical deterministic approximation methods (such as finite differences or finite elements in the context of PDEs; cf., e.g., Jovanović & Süli [17] and Tadmor [30]), as such classical approximations suffer under the *curse of dimensionality* in the sense that the amount of parameters to describe such approximations grows at least exponentially in the input dimension (cf., e.g., Bellman [2], Novak & Woźniakowski [20, Chapter 1], and Novak & Woźniakowski [21, Section 9.7]).

In many cases numerical simulations for ANNs suggest that ANN approximations are capable of approximating such extremely high-dimensional input-output data relationships and, in particular, numerical simulations suggest that ANN approximations for such problems seem to overcome the curse of dimensionality in the sense that the amount of real numbers used to describe those approximations seems to grow at most polynomially in the reciprocal  $\varepsilon^{-1}$ of the approximation precision  $\varepsilon > 0$  and the dimension  $d \in \mathbb{N}$  of the domain of the target function of the considered learning problem. In the information based complexity (IBC) literature this polynomial growth estimate in both the reciprocal of the approximation precision and the problem dimension is also often referred to as *polynomial tractability* (cf., e.g., Novak & Woźniakowski [20, Chapter 1] and Novak & Woźniakowski [21, Section 9.7]).

In the most simple form, an ANN describes a function (the so-called realization function of the ANN) which is given by iterated compositions of affine linear functions (with the entries of the multiplicative matrix in the affine linear function and the entries of the additive vector in the affine linear function described through a parameter vector of the ANN) and certain fixed nonlinear functions (referred to as activation functions). Roughly speaking, the result of such iterated composition after each nonlinear function represents a hidden layer of the ANN and ANNs with one (or none) hidden layers are referred to as shallow ANNs while ANNs with two or more hidden layers are called deep ANNs with the number of hidden layers representing the depth of the ANN (see also Figure 1 below for a graphical illustration of the architecture of an ANN).

Succesfull implementations in the above named learning problems usually employ deep

ANNs with a large number of hidden layers. In particular, the modern language processing framework BERT (Bidirectional Encoder Representations from Transformers) introduced in Devlin et al. [9] set new benchmarks in several natural language processing tasks like GLUE (standing for General Language Understanding Evaluation; see Wang et al. [34]) and MultiNLI (standing for Multi-Genre Natural Language Inference; see Williams et al. [35]) using ANNs with 11 and 23 hidden layers. The methods described in He et al. [15] won several image recognition competitions in 2015 like ILSVRC (standing for ImageNet Large Scale Visual Recognition Challenge; see Russakovsky et al. [26]) and MS COCO (standing for Microsoft Common Objects in Context; see Tsung-Yiet et al. [18]) by successfully implementing and training residual ANNs with 150 hidden layers and in 2017 the DenseNets in Huang et al. [16] consisting of 38 to 248 hidden layers outperformed state of the art techniques in visual object recognition.

The large number of numerical simulations in the above named learning problems also indicate that shallow or insufficiently deep ANNs might not be able to approximate the considered high-dimensional target functions without the curse of dimensionality. Taking this into account, a natural topic of research is to develop a mathematical theory which rigorosly explains why (and for which classes of target functions) deep ANNs seem to be capable of overcoming the curse of dimensionality while shallow or insufficiently deep ANNs seem to fail to do so in many relevant learning problems. In the scientific literature there are also a few mathematical research articles which contribute or have strong connections to this area of research.

In particular, we refer to Daniely [8] for a class of functions which can be approximated without the curse of dimensionality by ANNs with two hidden layers but not by shallow ANNs in a suitable class of non-standard ANNs with the multiplicative matrices in the affine linear transformations of the ANNs being suitable block matrices, we refer to Chui et al. [7] for classes of radial-basis functions which can be approximated without the curse of dimensionality by ANNs with certain smooth bounded sigmoidal activation functions if they have two hidden layers but not if they are shallow, we refer to Eldan & Shamir [11] for a sequence of two hidden layer ANNs (with the number of parameters growing at most polynomially in the input dimension) which cannot be approximated by shallow ANNs without the curse of dimensionality (teacher-student setup; cf., e.g., Saad & Solla [27] and Riegler & Biel [25]), and we refer to Venturi et al. [33] for a family of oscillating complex-valued functions which can be approximated in the  $L^2$ -sense with respect to a certain absolutely continuous probability measure without the curse of dimensionality by ANNs with two hidden layers but not by shallow ANNs.

We refer to Telgarsky [31, 32] and Yu et al. [36] for suitable families of *deep ANNs* indexed over an external parameter with at most polynomially many ANN parameters (with respect to the external parameter) which can only be approximated by *insufficiently deep ANNs* if the number of ANN parameters in the insufficiently deep ANNs grows at least exponentially in the external parameter (teacher-student setup; cf., e.g., Saad & Solla [27] and Riegler & Biel [25]) and we refer to Liang & Wu [4] for families of functions whose Fourier transformations can be approximated on cubes by deep ANNs with the number of parameters growing at most logarithmically in the length of the edges of the cubes but which can only be approximated on cubes by insufficiently deep ANNs with the number of parameters growing at least linearly in the length of the edges of the cubes. We refer to Safran & Shamir [28] for families of twice continuously differentiable functions whose approximating ANNs with a fixed depth require an amount of parameters which grows at least polynomially in the reciprocal of the approximation precision while the same accuracy can be achieved by deep ANNs with the depth and the total amount of parameters growing at most polylogarithmically in the reciprocal of the approximation precision. We refer to Grohs et al. [13] for a family of continuous functions which can be approximated by deep ANNs with the number of parameters growing at most cubically in the input dimension while the approximation with insufficiently deep ANNs suffers from the curse of dimensionality. Even though the results in Grohs et al. [13] show that certain target functions can be approximated without the curse of dimensionality by deep ANNs but not by insufficiently deep ANNs, the exponential growth of the amount of parameters in the insufficiently deep ANNs might not be very surprising as the target functions themselves in Grohs et al. [13] grow exponentially in the input dimension. In addition, we note that the approximation error in Grohs et al. [13] is measured via the  $L^2$ -norm with respect to the standard normal distribution on the whole space (instead of, say, the  $L^{\infty}$ -norm with respect to the Lebesgue measure on a d-dimensional cube). It remains an open problem to prove or disprove the conjecture that such phenomena also occur for target functions which are at most polynomially growing in the input dimension of the considered learning problem as it is usually the case in applications.

It is a key contribution of this article to answer this question affirmatively by explicitly revealing a sequence of at most polynomially growing simple functions which can be approximated without the curse of dimensionality by deep ANNs but cannot be approximated without the curse of dimensionality by shallow or insufficiently deep ANNs. In particular, we prove that there exist classes of simple uniformly globally bounded infinitely often differentiable functions which can be approximated without the curse of dimensionality by deep ANNs. In particular, we prove that there exist classes of simple uniformly globally bounded infinitely often differentiable functions which can be approximated without the curse of dimensionality by deep ANNs even if the absolute values of the ANN parameters are bounded by 1, but which cannot be approximated without the curse of dimensionality by shallow or insufficiently deep ANNs even if the ANN parameters may be arbitrarily large (see Theorem 1.3 below and its extensions in Theorem 5.2, Corollary 5.3, and Theorem 5.9 in Section 5 below). This is particularly relevant as the number and the size of the real valued parameters in the approximating ANN are direct indicators for the amount of memory needed to store the ANN on a computer and are, thereby, directly linked to the amount of memory needed on a computer to store a solution of the approximation problem.

To illustrate the findings of this work in more details, we now depict in this introductory section three representative key ANN approximation results of this article in a precise and selfcontained way (see Theorem 1.2, Theorem 1.3, and Theorem 1.4 below). Each of these three ANN approximation results employs the mathematical description of standard fully-connected feedforward ANNs with the rectified linear unit (ReLU) activation which is the subject of the following mathematical framework; see (1.1), (1.2), and (1.3) in Setting 1.1 below. We also refer to Figure 1 for a graphical illustration of the architecture of the ANNs formulated in Setting 1.1.



Figure 1: Graphical illustration of the architecture of the ANNs used in (1.1) and (1.2) in Setting 1.1: ANNs with L + 1 layers (with L affine linear transformations) with an  $l_0$ -dimensional input layer ( $l_0$  neurons on the input layer), an  $l_1$ -dimensional 1<sup>st</sup> hidden layer ( $l_1$  neurons on the 1<sup>st</sup> hidden layer), an  $l_2$ -dimensional 2<sup>nd</sup> hidden layer ( $l_2$  neurons on the 2<sup>nd</sup> hidden layer), ..., an  $l_{L-1}$ -dimensional (L - 1)<sup>th</sup> hidden layer ( $l_{L-1}$  neurons on the (L - 1)<sup>th</sup> hidden layer), and an  $l_L$ -dimensional output layer ( $l_L$  neurons on the output layer). The realization function in (1.2) in Setting 1.1 assignes the  $l_0$ -dimensional input vector  $x_0 = (x_{0,1}, \ldots, x_{0,l_0}) \in \mathbb{R}^{l_0}$  to the  $l_1$ -dimensional vector  $x_1 = (x_{1,1}, \ldots, x_{1,l_1}) \in \mathbb{R}^{l_1}$  with  $x_1 = \Re(W_1 x_0 + B_1)$  on the 1<sup>st</sup> hidden layer, assignes the vector  $x_1 = (x_{1,1}, \ldots, x_{1,l_1}) \in \mathbb{R}^{l_1}$  on the 1<sup>st</sup> hidden layer to the vector  $x_2 = (x_{2,1}, \ldots, x_{2,l_2}) \in \mathbb{R}^{l_2}$  with  $x_2 = \Re(W_2 x_1 + B_2)$  on the 2<sup>nd</sup> hidden layer, ..., assignes the vector  $x_{L-2} = (x_{L-2,1}, \ldots, x_{L-2,l_{L-2}}) \in \mathbb{R}^{l_{L-2}}$  on the (L - 2)<sup>th</sup> hidden layer to the vector  $x_{L-1} = (x_{L-1,1}, \ldots, x_{L-1,l_{L-1}}) \in \mathbb{R}^{l_{L-1}}$  with  $x_{L-1} = \Re(W_{L-1} x_{L-2} + B_{L-1})$  on the (L - 1)<sup>th</sup> hidden layer, and assignes the vector  $x_{L-1} = (x_{L-1,1}, \ldots, x_{L-1,l_{L-1}}) \in \mathbb{R}^{l_{L-1}}$  with  $x_L = W_L x_{L-1} + B_L$  on the output layer.

Setting 1.1. Let  $\mathfrak{R}: (\bigcup_{k \in \mathbb{N}} \mathbb{R}^k) \to (\bigcup_{k \in \mathbb{N}} \mathbb{R}^k)$  and  $|\cdot|: (\bigcup_{k,l \in \mathbb{N}} (\mathbb{R}^{k \times l} \times \mathbb{R}^k)) \to \mathbb{R}$  satisfy for all  $k, l \in \mathbb{N}, x = (x_1, \ldots, x_k) \in \mathbb{R}^k, W = (W_{i,j})_{(i,j) \in \{1, \ldots, k\} \times \{1, \ldots, l\}} \in \mathbb{R}^{k \times l}$  that

 $\Re(x) = (\max\{x_1, 0\}, \dots, \max\{x_k, 0\}) \text{ and } ||(W, x)|| = \max_{1 \le i \le k} \max_{1 \le j \le l} \max\{|W_{i,j}|, |x_i|\},$ (1.1)

let  $\mathbf{N} = \bigcup_{L \in \mathbb{N}} \bigcup_{l_0, l_1, \dots, l_L \in \mathbb{N}} (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})), \text{ let } \mathcal{R} \colon \mathbf{N} \to (\bigcup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l)), \mathcal{L} \colon \mathbf{N} \to \mathbb{N}, \mathcal{P} \colon \mathbf{N} \to \mathbb{N}, \text{ and } \|\cdot\| \colon \mathbf{N} \to \mathbb{R} \text{ satisfy for all } L \in \mathbb{N}, \ l_0, l_1, \dots, l_L \in \mathbb{N}, \ \mathcal{F} = ((W_1, B_1), \dots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})), \ x_0 \in \mathbb{R}^{l_0}, \ x_1 \in \mathbb{R}^{l_1}, \dots, \ x_{L-1} \in \mathbb{R}^{l_{L-1}} \text{ with } \forall k \in \mathbb{N} \cap (0, L) \colon x_k = \Re(W_k x_{k-1} + B_k) \text{ that}$ 

$$\mathcal{R}(\boldsymbol{\ell}) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_L}), \qquad (\mathcal{R}(\boldsymbol{\ell}))(x_0) = W_L x_{L-1} + B_L, \qquad \mathcal{L}(\boldsymbol{\ell}) = L, \qquad (1.2)$$

$$\mathcal{P}(\mathbf{f}) = \sum_{k=1}^{L} l_k (l_{k-1} + 1), \quad and \quad \|\mathbf{f}\| = \max_{1 \le k \le L} \|(W_k, B_k)\|, \quad (1.3)$$

let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ , and let  $\text{Cost}: (\bigcup_{d \in \mathbb{N}} C(\mathbb{R}^d, \mathbb{R})) \times [0, \infty]^3 \to [0, \infty]$  satisfy for all  $d \in \mathbb{N}$ ,  $f \in C(\mathbb{R}^d, \mathbb{R})$ ,  $L, S, \varepsilon \in [0, \infty]$  that

$$\operatorname{Cost}(f, L, S, \varepsilon) = \min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix}\exists \not f \in \mathbb{N} \colon (\mathcal{P}(f) = p) \land (\mathcal{L}(f) \leq L) \land \\ (\|f\| \leq S) \land (\mathcal{R}(f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(f))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right). (1.4)$$

In Setting 1.1 we also introduce the cost-functional which we employ to state our ANN approximation results in Theorem 1.2, Theorem 1.3, and Theorem 1.4. Specifically, we note that (1.4) in Setting 1.1 ensures that for every dimension  $d \in \mathbb{N}$ , every continuous function  $f: \mathbb{R}^d \to \mathbb{R}$ , every upper bound  $L \in [0, \infty]$  for the depths of the ANNs, every upper bound  $S \in [0, \infty]$  for the size of the absolute values of the ANN parameters, and every approximation precision  $\varepsilon \in [0, \infty]$  we have that  $\operatorname{Cost}(f, L, S, \varepsilon)$  represents the minimal amount of ANN parameters needed to approximate the target function  $f: \mathbb{R}^d \to \mathbb{R}$  with the error tolerance  $\varepsilon$ among the class of ANNs with at most L affine linear transformations and the absolute values of the ANN parameters beeing at most S. Using Setting 1.1 we now formulate the above mentioned three representative key ANN approximation results.

**Theorem 1.2.** Assume Setting 1.1. Then there exist  $\mathbf{c} \in (0, \infty)$  and infinitely often differentiable  $f_d \colon \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}$ , with compact support and  $\sup_{d \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |f_d(x)| \leq 1$  such that for all  $d, L \in \mathbb{N}, \varepsilon \in (0, 1)$  it holds that

$$\operatorname{Cost}(f_d, L, \infty, \varepsilon) \ge 2^{\frac{a}{L}} \quad and \quad \operatorname{Cost}(f_d, \mathfrak{c}d, 1, \varepsilon) \le \mathfrak{c}d^2\varepsilon^{-2}.$$
(1.5)

Theorem 1.2 above is an immediate consequence of Corollary 5.10 in Section 5 below and Corollary 5.10, in turn, follows from Theorem 5.9 in Section 5. Roughly speaking, Theorem 1.2 reveals that there exists a sequence of smooth and uniformly globally bounded functions  $f_d \colon \mathbb{R}^d \to \mathbb{R}$  for  $d \in \mathbb{N}$  with compact support which can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs even if the ANN parameters may be arbitrarily large (see the first inequality in (1.5)) but which can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1 (see the second inequality in (1.5)). Theorem 1.2 only asserts the existence of suitable smooth and uniformly globally bounded target functions which can be approximated without the curse of dimensionality by deep ANNs but not by insufficiently deep ANNs but it does not explicitly specify the employed target functions. However, in the more general approximation result in Theorem 5.9 in Section 5 we also explicitly specify a class of simple target functions which we use to prove Theorem 1.2. In addition, in this work we also reveal that the sine of the product functions serve as one possible sequence of smooth and uniformly globally bounded target functions for which the conclusion of Theorem 1.2 essentially applies and this is precisely the subject of our next representative key ANN approximation result.

**Theorem 1.3.** Assume Setting 1.1, assume  $b - a \ge 7$ , and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \sin(\prod_{i=1}^d x_i)$ . Then there exists  $\mathfrak{c} \in (0, \infty)$ such that for all  $d, L \in \mathbb{N}, \varepsilon \in (0, 1)$  it holds that

$$\operatorname{Cost}(f_d, L, \infty, \varepsilon) \ge 2^{\frac{d}{L}} \quad and \quad \operatorname{Cost}(f_d, \mathfrak{c} d^2 \varepsilon^{-1}, 1, \varepsilon) \le \mathfrak{c} d^3 \varepsilon^{-2}.$$
(1.6)

Theorem 1.3 above is an immediate consequence of Theorem 5.2 in Section 5 below. Theorem 1.3 shows that the sine of the product functions can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs even if the ANN parameters may be arbitrarily large (see the first inequality in (1.6)) but can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1 (see the second inequality in (1.6)). Actually, a bit modified and somehow weakened variant of the conclusion of Theorem 1.3 applies also to the product functions themselves. This is precisely the subject of our final representative key ANN approximation result in this introductory section.

**Theorem 1.4.** Assume Setting 1.1, assume  $b - a \ge 4$ , and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \prod_{i=1}^d x_i$ . Then there exists  $\mathfrak{c} \in (0, \infty)$  such that for all  $c \in [\mathfrak{c}, \infty)$ ,  $d, L \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  it holds that

$$\operatorname{Cost}(f_d, L, cd^c, \varepsilon) \ge \left[ (4cL)^{-3c} \right] 2^{\frac{d}{2L}} \quad and \quad \operatorname{Cost}(f_d, cd^2 \varepsilon^{-1}, 1, \varepsilon) \le cd^3 \varepsilon^{-1}.$$
(1.7)

Theorem 1.4 above is an immediate consequence of Theorem 5.1 in Section 5 below. Loosely speaking, Theorem 1.4 proves that the plane vanilla product functions can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs if the absolute values of the ANN parameters are polynomially bounded in the input dimension  $d \in \mathbb{N}$  (see the first inequality in (1.7)) but can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1 (see the second inequality in (1.7)).

The remainder of this article is organized as follows. In Section 2 we present the concepts, operations, and elementary preparatory results regarding ANNs that we frequently employ in Sections 3, 4, and 5. In Section 3 we establish suitable lower bounds for the minimal number of parameters of shallow or insufficiently deep ANNs to approximate certain high-dimensional target functions. In Section 4 we establish suitable upper bounds for the minimal number of parameters of ANNs to approximate the product functions and certain highly oscillating functions in the case where the absolute values of the parameters of the ANNs are assumed to be uniformly bounded by 1. In Section 5 we combine the main results from Section 3 and Section 4 to obtain that the product functions and certain highly oscillating functions can

be approximated without the curse of dimensionality by deep ANNs but not by insufficiently deep ANNs and, thereby, we prove our three representative key ANN approximation results in Theorem 1.2, Theorem 1.3, and Theorem 1.4 above.

## 2 Artificial neural network (ANN) calculus

The purpose of this section is to introduce the concepts, operations, and elementary preparatory results regarding ANNs that we frequently employ in the later sections of this article.

In particular, in Definition 2.1 in Section 2.1 we recall the notion of the set of ANNs N in the structured description as well as several basic functions acting on this set of ANNs such as the parameter function  $\mathcal{P} \colon \mathbf{N} \to \mathbb{N}$  for ANNs (counting the number of parameters of an ANN), the length function  $\mathcal{L} \colon \mathbf{N} \to \mathbb{N}$  for ANNs (measuring the number of affine linear transformations of an ANN), the input dimension function  $\mathcal{I} \colon \mathbf{N} \to \mathbb{N}$  for ANNs (specifying the number of neurons on the input layer of an ANN), the output dimension function  $\mathcal{O} \colon \mathbf{N} \to \mathbb{N}$  for ANNs (specifying the number of neurons on the output layer of an ANN), the hidden layer function  $\mathcal{H} \colon \mathbf{N} \to \mathbb{N}_0$  for ANNs (counting the number of hidden layers of an ANN), the layer dimension vector function  $\mathcal{D} \colon \mathbf{N} \to (\cup_{L \in \mathbb{N}} \mathbb{N}^L)$  for ANNs (representing the numbers of neurons on the layers of an ANN), as a vector), and the layer dimension functions  $\mathbb{D}_n \colon \mathbf{N} \to \mathbb{N}_0$ ,  $n \in \mathbb{N}_0$ , for ANNs (counting the numbers of neurons on the layers of an ANN).

In Section 2.2 we recall the concept of realization functions of ANNs, in Section 2.3 we recall the concept of parallelizations of ANNs, in Section 2.4 we recall suitable ANNs whose realization functions exactly coincide with the real identity functions, in Section 2.5 we recall the concept of compositions of ANNs, and in Section 2.6 we present elementary concepts and results regarding the sizes of the absolute values of the parameters of ANNs.

Most of the concepts and results in this section are well known and have appeared, often in a bit different form, in previous works in the literature (cf., e.g., [1, 3, 6, 10, 12–14, 23]). In particular, Definition 2.1 is a slightly extended version of, e.g., Grohs et al. [12, Definition 2.1], Definition 2.2 corresponds to, e.g., Grohs et al. [13, Definition 2.1], Definition 2.3 is a reformulated variant of, e.g., Grohs et al. [12, Definition 2.3], Definition 2.4 is a reformulated variant of, e.g., Grohs et al. [12, Definition 2.17], Proposition 2.5 is a slightly differently presented variant of, e.g., Grohs et al. [12, Lemma 2.18 and Proposition 2.19], Definition 2.6 corresponds to, e.g., Grohs et al. [13, Definition 2.13], Proposition 2.7 corresponds to, e.g., Grohs et al. [13, Proposition 2.14], Definition 2.8 is a reformulated variant of, e.g., Grohs et al. [12, Definition 2.5], Lemma 2.9 corresponds to, e.g., Grohs et al. [12, Lemma 2.8], Proposition 2.10 corresponds to, e.g., Beneventano et al. [3, Lemma 2.16], Lemma 2.11 corresponds to, e.g., Beneventano et al. [3, Lemma 2.17], (2.15) in Definition 2.13 is a slightly differently presented variant of, e.g., Grohs et al. [13, Definitions 2.21 and 2.22], and item (i) in Lemma 2.16 is a reformulated special case of, e.g., Grohs et al. [13, Lemma 2.23].

#### 2.1 Set of ANNs

**Definition 2.1** (Set of ANNs). We denote by **N** the set given by

$$\mathbf{N} = \bigcup_{L \in \mathbb{N}} \bigcup_{l_0, l_1, \dots, l_L \in \mathbb{N}} \left( \mathbf{X}_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}) \right),$$
(2.1)

we denote by  $\mathcal{P}: \mathbf{N} \to \mathbb{N}, \mathcal{L}: \mathbf{N} \to \mathbb{N}, \mathcal{I}: \mathbf{N} \to \mathbb{N}, \mathcal{O}: \mathbf{N} \to \mathbb{N}, \mathcal{H}: \mathbf{N} \to \mathbb{N}_0, \text{ and } \mathcal{D}: \mathbf{N} \to (\cup_{L \in \mathbb{N}} \mathbb{N}^L)$  the functions which satisfy for all  $L \in \mathbb{N}, l_0, l_1, \ldots, l_L \in \mathbb{N}, f \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$  that

$$\mathcal{P}(\mathbf{f}) = \sum_{k=1}^{L} l_k (l_{k-1} + 1), \quad \mathcal{L}(\mathbf{f}) = L, \quad \mathcal{I}(\mathbf{f}) = l_0, \quad \mathcal{O}(\mathbf{f}) = l_L, \quad \mathcal{H}(\mathbf{f}) = L - 1, \quad (2.2)$$

and  $\mathcal{D}(\mathscr{L}) = (l_0, l_1, \ldots, l_L)$ , for every  $n \in \mathbb{N}_0$  we denote by  $\mathbb{D}_n \colon \mathbf{N} \to \mathbb{N}_0$  the function which satisfies for all  $L \in \mathbb{N}, l_0, l_1, \ldots, l_L \in \mathbb{N}, \ \mathscr{L} \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$  that

$$\mathbb{D}_n(\mathscr{L}) = \begin{cases} l_n & : n \le L\\ 0 & : n > L, \end{cases}$$
(2.3)

and for every  $L \in \mathbb{N}$ ,  $l_0, l_1, \ldots, l_L \in \mathbb{N}$ ,  $\ell = ((W_1, B_1), \ldots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$  we denote by  $\mathcal{W}_{(\cdot),\ell} = (\mathcal{W}_{n,\ell})_{n \in \{1,2,\ldots,L\}} \colon \{1,2,\ldots,L\} \to (\cup_{k,m \in \mathbb{N}} \mathbb{R}^{k \times m})$  and  $\mathcal{B}_{(\cdot),\ell} = (\mathcal{B}_{n,\ell})_{n \in \{1,2,\ldots,L\}} \colon \{1,2,\ldots,L\} \to (\cup_{k \in \mathbb{N}} \mathbb{R}^k)$  the functions which satisfy for all  $n \in \{1,2,\ldots,L\}$  that  $\mathcal{W}_{n,\ell} = W_n$  and  $\mathcal{B}_{n,\ell} = B_n$ .

#### 2.2 Realization functions of ANNs

**Definition 2.2** (Multidimensional ReLU). We denote by  $\mathfrak{R}: (\bigcup_{d \in \mathbb{N}} \mathbb{R}^d) \to (\bigcup_{d \in \mathbb{N}} \mathbb{R}^d)$  the function which satisfies for all  $d \in \mathbb{N}, x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$\Re(x) = (\max\{x_1, 0\}, \max\{x_2, 0\}, \dots, \max\{x_d, 0\}).$$
(2.4)

**Definition 2.3** (Realization functions of ANNs). We denote by  $\mathcal{R} \colon \mathbf{N} \to (\bigcup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l))$ the function which satisfies for all  $\boldsymbol{\ell} \in \mathbf{N}, x_0 \in \mathbb{R}^{\mathbb{D}_0(\boldsymbol{\ell})}, x_1 \in \mathbb{R}^{\mathbb{D}_1(\boldsymbol{\ell})}, \dots, x_{\mathcal{H}(\boldsymbol{\ell})} \in \mathbb{R}^{\mathbb{D}_{\mathcal{H}(\boldsymbol{\ell})}(\boldsymbol{\ell})}$  with  $\forall k \in \mathbb{N} \cap [0, \mathcal{H}(\boldsymbol{\ell})] \colon x_k = \mathfrak{R}(\mathcal{W}_{k, \boldsymbol{\ell}} x_{k-1} + \mathcal{B}_{k, \boldsymbol{\ell}})$  that

$$\mathcal{R}(\boldsymbol{\ell}) \in C(\mathbb{R}^{\mathcal{I}(\boldsymbol{\ell})}, \mathbb{R}^{\mathcal{O}(\boldsymbol{\ell})}) \quad \text{and} \quad (\mathcal{R}(\boldsymbol{\ell}))(x_0) = \mathcal{W}_{\mathcal{L}(\boldsymbol{\ell}), \boldsymbol{\ell}} x_{\mathcal{H}(\boldsymbol{\ell})} + \mathcal{B}_{\mathcal{L}(\boldsymbol{\ell}), \boldsymbol{\ell}}$$
(2.5)

(cf. Definitions 2.1 and 2.2).

### 2.3 Parallelizations of ANNs

**Definition 2.4** (Parallelization of ANNs). Let  $n \in \mathbb{N}$ . Then we denote by

$$\mathbf{P}_n: \left\{ \boldsymbol{\ell} = (\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_n) \in \mathbf{N}^n: \mathcal{L}(\boldsymbol{\ell}_1) = \mathcal{L}(\boldsymbol{\ell}_2) = \dots = \mathcal{L}(\boldsymbol{\ell}_n) \right\} \to \mathbf{N}$$
(2.6)

the function which satisfies for all  $\boldsymbol{\ell} = (\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_n) \in \mathbf{N}^n, k \in \{1, 2, \dots, \mathcal{L}(\boldsymbol{\ell}_1)\}$  with  $\mathcal{L}(\boldsymbol{\ell}_1) = \mathcal{L}(\boldsymbol{\ell}_2) = \dots = \mathcal{L}(\boldsymbol{\ell}_n)$  that

$$\mathcal{L}(\mathbf{P}_{n}(\boldsymbol{\ell})) = \mathcal{L}(\boldsymbol{\ell}_{1}), \ \mathcal{W}_{k,\mathbf{P}_{n}(\boldsymbol{\ell})} = \begin{pmatrix} \mathcal{W}_{k,\boldsymbol{\ell}_{1}} & 0 & \cdots & 0 \\ 0 & \mathcal{W}_{k,\boldsymbol{\ell}_{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_{k,\boldsymbol{\ell}_{n}} \end{pmatrix}, \text{ and } \mathcal{B}_{k,\mathbf{P}_{n}(\boldsymbol{\ell})} = \begin{pmatrix} \mathcal{B}_{k,\boldsymbol{\ell}_{1}} \\ \mathcal{B}_{k,\boldsymbol{\ell}_{2}} \\ \vdots \\ \mathcal{B}_{k,\boldsymbol{\ell}_{n}} \end{pmatrix}$$
(2.7)

(cf. Definition 2.1).

**Proposition 2.5.** Let  $n \in \mathbb{N}$ ,  $\mathcal{f} = (f_1, \ldots, f_n) \in \mathbb{N}^n$  satisfy  $\mathcal{L}(f_1) = \mathcal{L}(f_2) = \ldots = \mathcal{L}(f_n)$  (cf. Definition 2.1). Then

- (i) it holds for all  $k \in \mathbb{N}_0$  that  $\mathbb{D}_k(\mathbf{P}_n(\mathbf{\ell})) = \sum_{j=1}^n \mathbb{D}_k(\mathbf{\ell}_j)$ ,
- (*ii*) it holds that  $\mathcal{R}(\mathbf{P}_n(\mathbf{f})) \in C(\mathbb{R}^{[\sum_{j=1}^n \mathcal{I}(\mathbf{f}_j)]}, \mathbb{R}^{[\sum_{j=1}^n \mathcal{O}(\mathbf{f}_j)]})$ , and

(iii) it holds for all  $x_1 \in \mathbb{R}^{\mathcal{I}(\ell_1)}, x_2 \in \mathbb{R}^{\mathcal{I}(\ell_2)}, \dots, x_n \in \mathbb{R}^{\mathcal{I}(\ell_n)}$  that

$$\left(\mathcal{R}\big(\mathbf{P}_n(\boldsymbol{\ell})\big)\big)(x_1, x_2, \dots, x_n) = \left((\mathcal{R}(\boldsymbol{\ell}_1))(x_1), (\mathcal{R}(\boldsymbol{\ell}_2))(x_2), \dots, (\mathcal{R}(\boldsymbol{\ell}_n))(x_n)\right)$$
(2.8)

(cf. Definitions 2.3 and 2.4).

Proof of Proposition 2.5. Observe that, e.g., Grohs et al. [12, Lemma 2.18] establishes item (i). Note that, e.g., Grohs et al. [12, Proposition 2.19] demonstrates items (ii) and (iii). The proof of Proposition 2.5 is thus complete.

#### 2.4 Identity ANNs

**Definition 2.6** (Identity ANNs). We denote by  $(\mathbb{I}_d)_{d\in\mathbb{N}} \subseteq \mathbf{N}$  the ANNs which satisfy for all  $d \in \mathbb{N} \cap [2, \infty)$  that

$$\mathbb{I}_{1} = \left( \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \left( \begin{pmatrix} 1 & -1 \end{pmatrix}, 0 \right) \right) \in \left( (\mathbb{R}^{2 \times 1} \times \mathbb{R}^{2}) \times (\mathbb{R}^{1 \times 2} \times \mathbb{R}^{1}) \right)$$
(2.9)

and  $\mathbb{I}_d = \mathbf{P}_d(\mathbb{I}_1, \mathbb{I}_1, \dots, \mathbb{I}_1)$  (cf. Definitions 2.1 and 2.4).

**Proposition 2.7.** Let  $d \in \mathbb{N}$ . Then

- (i) it holds that  $\mathcal{R}(\mathbb{I}_d) = \mathrm{id}_{\mathbb{R}^d}$ ,
- (ii) it holds that  $\mathcal{D}(\mathbb{I}_d) = (d, 2d, d)$ , and
- (iii) it holds that  $\mathcal{P}(\mathbb{I}_d) = 4d^2 + 3d$
- (cf. Definitions 2.1, 2.3, and 2.6).

*Proof of Proposition 2.7.* Observe that, e.g., Grohs et al. [13, Proposition 2.14] establishes items (i), (ii), and (iii). The proof of Proposition 2.7 is thus complete.  $\Box$ 

#### 2.5 Compositions of ANNs

**Definition 2.8** (Compositions of ANNs). We denote by  $(\cdot) \bullet (\cdot)$ : { $\not f = (\not f_1, \not f_2) \in \mathbb{N} \times \mathbb{N}$ :  $\mathcal{I}(\not f_1) = \mathcal{O}(\not f_2)$ }  $\rightarrow \mathbb{N}$  the function which satisfies for all  $L, \mathfrak{L} \in \mathbb{N}, \not f_1, \not f_2 \in \mathbb{N}, k \in \mathbb{N} \cap (0, L + \mathfrak{L})$  with  $\mathcal{I}(\not f_1) = \mathcal{O}(\not f_2), \mathcal{L}(\not f_2) = L$ , and  $\mathcal{L}(\not f_1) = \mathfrak{L}$  that  $\mathcal{L}(\not f_1 \bullet \not f_2) = L + \mathfrak{L} - 1$  and

$$\left(\mathcal{W}_{k,\ell_{1}\bullet\ell_{2}},\mathcal{B}_{k,\ell_{1}\bullet\ell_{2}}\right) = \begin{cases} \left(\mathcal{W}_{k,\ell_{2}},\mathcal{B}_{k,\ell_{2}}\right) & :k < L\\ \left(\mathcal{W}_{1,\ell_{1}}\mathcal{W}_{L,\ell_{2}},\mathcal{W}_{1,\ell_{1}}\mathcal{B}_{L,\ell_{2}} + \mathcal{B}_{1,\ell_{1}}\right) & :k = L\\ \left(\mathcal{W}_{k-L+1,\ell_{1}},\mathcal{B}_{k-L+1,\ell_{1}}\right) & :k > L \end{cases}$$
(2.10)

(cf. Definition 2.1).

**Lemma 2.9.** Let  $f_1, f_2, f_3 \in \mathbb{N}$  satisfy  $\mathcal{I}(f_1) = \mathcal{O}(f_2)$  and  $\mathcal{I}(f_2) = \mathcal{O}(f_3)$  (cf. Definition 2.1). Then  $(f_1 \bullet f_2) \bullet f_3 = f_1 \bullet (f_2 \bullet f_3)$  (cf. Definition 2.8).

*Proof of Lemma 2.9.* Note that, e.g., Grohs et al. [12, Lemma 2.8] shows  $(\not_1 \bullet \not_2) \bullet \not_3 = \not_1 \bullet (\not_2 \bullet \not_3)$ . The proof of Lemma 2.9 is thus complete.

**Proposition 2.10.** Let  $n \in \mathbb{N} \cap (1, \infty)$ ,  $\not{\ell}_1, \not{\ell}_2, \ldots, \not{\ell}_n \in \mathbb{N}$  satisfy for all  $k \in \mathbb{N} \cap (1, n]$  that  $\mathcal{I}(\not{\ell}_{k-1}) = \mathcal{O}(\not{\ell}_k)$  (cf. Definition 2.1). Then

- (i) it holds that  $\mathcal{R}(\mathcal{F}_1 \bullet \mathcal{F}_2 \bullet \ldots \bullet \mathcal{F}_n) = [\mathcal{R}(\mathcal{F}_1)] \circ [\mathcal{R}(\mathcal{F}_2)] \circ \ldots \circ [\mathcal{R}(\mathcal{F}_n)]$  and
- (ii) it holds that  $\mathcal{H}(\mathcal{F}_1 \bullet \mathcal{F}_2 \bullet \ldots \bullet \mathcal{F}_n) = \sum_{k=1}^n \mathcal{H}(\mathcal{F}_k)$

(cf. Definitions 2.3 and 2.8 and Lemma 2.9).

*Proof of Proposition 2.10.* Note that, e.g., Beneventano et al. [3, Proposition 2.16] (see, e.g., also Grohs et al. [12, Proposition 2.6]) establishes items (i) and (ii). The proof of Proposition 2.10 is thus complete.  $\Box$ 

**Lemma 2.11.** Let  $f, g \in \mathbb{N}$  satisfy  $\mathcal{I}(f) = \mathcal{O}(g)$  (cf. Definition 2.1). Then

(i) it holds that

$$\mathcal{D}(\boldsymbol{f} \bullet \boldsymbol{g}) = (\mathbb{D}_0(\boldsymbol{g}), \mathbb{D}_1(\boldsymbol{g}), \dots, \mathbb{D}_{\mathcal{H}(\boldsymbol{g})}(\boldsymbol{g}), \mathbb{D}_1(\boldsymbol{f}), \mathbb{D}_2(\boldsymbol{f}), \dots, \mathbb{D}_{\mathcal{L}(\boldsymbol{f})}(\boldsymbol{f}))$$
(2.11)

and

(ii) it holds that

$$\mathcal{D}(\boldsymbol{f} \bullet \mathbb{I}_{\mathcal{O}(\boldsymbol{g})} \bullet \boldsymbol{g}) = (\mathbb{D}_{0}(\boldsymbol{g}), \mathbb{D}_{1}(\boldsymbol{g}), \dots, \mathbb{D}_{\mathcal{H}(\boldsymbol{g})}(\boldsymbol{g}), 2\mathbb{D}_{\mathcal{L}(\boldsymbol{g})}(\boldsymbol{g}), \mathbb{D}_{1}(\boldsymbol{f}), \mathbb{D}_{2}(\boldsymbol{f}), \dots, \mathbb{D}_{\mathcal{L}(\boldsymbol{f})}(\boldsymbol{f}))$$
(2.12)

(cf. Definitions 2.6 and 2.8).

*Proof of Lemma 2.11.* Note that, e.g., Beneventano et al. [3, Lemma 2.17] (see, e.g., also Grohs et al. [12, Proposition 2.6]) establishes items (i) and (ii). The proof of Lemma 2.11 is thus complete.  $\Box$ 

#### 2.6 Sizes of parameters of ANNs

**Definition 2.12** (Supremum norm). We denote by  $\|\cdot\|_{\infty}$ :  $(\bigcup_{k,l\in\mathbb{N}}\mathbb{R}^{k\times l}) \to \mathbb{R}$  the function which satisfies for all  $k, l \in \mathbb{N}, W = (W_{i,j})_{(i,j)\in\{1,2,\dots,k\}\times\{1,2,\dots,l\}} \in \mathbb{R}^{k\times l}$  that

$$||W||_{\infty} = \max_{i \in \{1, 2, \dots, k\}} \max_{j \in \{1, 2, \dots, l\}} |W_{i,j}|.$$
(2.13)

**Definition 2.13** (Sizes of parameters of ANNs). We denote by  $\mathbb{S}_r \colon \mathbb{N} \to \mathbb{R}$ ,  $r \in \{0, 1\}$ , the functions which satisfies for all  $r \in \{0, 1\}$ ,  $\ell \in \mathbb{N}$  that

$$\mathbb{S}_{r}(\boldsymbol{\ell}) = \max\{\|\mathcal{W}_{r\mathcal{H}(\boldsymbol{\ell})+1,\boldsymbol{\ell}}\|_{\infty}, \|\mathcal{B}_{r\mathcal{H}(\boldsymbol{\ell})+1,\boldsymbol{\ell}}\|_{\infty}\}$$
(2.14)

and we denote by  $\mathcal{S} \colon \mathbf{N} \to \mathbb{R}$  the function which satisfies for all  $\not \in \mathbf{N}$  that

$$\mathcal{S}(\boldsymbol{\ell}) = \max_{k \in \{1, 2, \dots, \mathcal{L}(\boldsymbol{\ell})\}} \max\{\|\mathcal{W}_{k, \boldsymbol{\ell}}\|_{\infty}, \|\mathcal{B}_{k, \boldsymbol{\ell}}\|_{\infty}\}$$
(2.15)

(cf. Definition 2.12).

**Lemma 2.14** (Sizes of ANN parameters of parallelizations). Let  $n \in \mathbb{N}$ ,  $f_1, f_2, \ldots, f_n \in \mathbb{N}$ satisfy  $\mathcal{L}(f_1) = \mathcal{L}(f_2) = \ldots = \mathcal{L}(f_n)$  (cf. Definition 2.1). Then

- (i) it holds that  $\mathcal{S}(\mathbf{P}_n(\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n)) = \max\{\mathcal{S}(\mathbf{f}_1), \mathcal{S}(\mathbf{f}_2), \dots, \mathcal{S}(\mathbf{f}_n)\}$  and
- (*ii*) it holds for all  $r \in \{0,1\}$  that  $\mathbb{S}_r(\mathbb{P}_n(\mathbb{P}_1,\mathbb{P}_2,\ldots,\mathbb{P}_n)) = \max\{\mathbb{S}_r(\mathbb{P}_1),\mathbb{S}_r(\mathbb{P}_2),\ldots,\mathbb{S}_r(\mathbb{P}_n)\}$

(cf. Definitions 2.4 and 2.13).

*Proof of Lemma 2.14.* Observe that (2.7) establishes items (i) and (ii). The proof of Lemma 2.14 is thus complete.

**Corollary 2.15** (Sizes of identity ANNs). Let  $d \in \mathbb{N}$ . Then  $\mathcal{S}(\mathbb{I}_d) = \mathbb{S}_0(\mathbb{I}_d) = \mathbb{S}_1(\mathbb{I}_d) = 1$  (cf. Definitions 2.6 and 2.13).

Proof of Corollary 2.15. Note that (2.9) and Lemma 2.14 establish  $\mathcal{S}(\mathbb{I}_d) = \mathbb{S}_0(\mathbb{I}_d) = \mathbb{S}_1(\mathbb{I}_d) = 1$ . The proof of Corollary 2.15 is thus complete.

**Lemma 2.16** (Sizes of ANN parameters of compositions). Let  $d \in \mathbb{N}$ . Then

(i) it holds for all  $\not e, g \in \mathbf{N}$  with  $\mathcal{I}(\not e) = \mathcal{O}(g)$  that

$$\mathcal{S}(\not e \bullet g) \le \max\{\mathcal{S}(\not e), \mathcal{S}(g), \mathbb{S}_0(\not e) \mathbb{S}_1(g)d + \mathbb{S}_0(\not e)\}$$
(2.16)

and

(ii) it holds for all 
$$r \in \{0,1\}$$
,  $\ell_0, \ell_1 \in \mathbb{N}$  with  $\mathcal{I}(\ell_1) = \mathcal{O}(\ell_0)$  and  $\mathcal{L}(\ell_r) > 1$  that

$$\mathbb{S}_r(\mathbf{\ell}_1 \bullet \mathbf{\ell}_0) = \mathbb{S}_r(\mathbf{\ell}_r) \tag{2.17}$$

(cf. Definitions 2.1, 2.8, and 2.13).

Proof of Lemma 2.16. Observe (2.13) implies that for all  $m, n \in \mathbb{N}, W \in \mathbb{R}^{m \times d}, B \in \mathbb{R}^m, \mathfrak{W} \in \mathbb{R}^{d \times n}, \mathfrak{B} \in \mathbb{R}^d$  it holds that

$$\|W\mathfrak{W}\|_{\infty} \le d\|W\|_{\infty}\|\mathfrak{W}\|_{\infty} \quad \text{and} \quad \|W\mathfrak{B} + B\|_{\infty} \le d\|W\|_{\infty}\|\mathfrak{B}\|_{\infty} + \|B\|_{\infty} \quad (2.18)$$

(cf. Definition 2.12). Combining this with (2.10) and (2.15) shows that for all  $\ell, q \in \mathbb{N}$  with  $\mathcal{I}(\ell) = \mathcal{O}(q)$  it holds that

$$\mathcal{S}(\boldsymbol{f} \bullet \boldsymbol{g}) \le \max\{\mathcal{S}(\boldsymbol{f}), \mathcal{S}(\boldsymbol{g}), \mathbb{S}_0(\boldsymbol{f}) \mathbb{S}_1(\boldsymbol{g})d + \mathbb{S}_0(\boldsymbol{f})\}$$
(2.19)

(cf. Definitions 2.1, 2.8, and 2.13). This establishes item (i). Note that (2.10) and (2.14) imply that for all  $r \in \{0, 1\}$ ,  $\ell_0, \ell_1 \in \mathbf{N}$  with  $\mathcal{I}(\ell_1) = \mathcal{O}(\ell_0)$  and  $\mathcal{L}(\ell_r) > 1$  that

$$\mathbb{S}_r(\mathbf{\ell}_1 \bullet \mathbf{\ell}_0) = \mathbb{S}_r(\mathbf{\ell}_r) \tag{2.20}$$

This establishes item (ii). The proof of Lemma 2.16 is thus complete.

**Proposition 2.17** (Sizes of ANN parameters of compositions). Let  $d \in \mathbb{N}$ ,  $f, g \in \mathbb{N}$  satisfy  $\mathcal{I}(f) = d = \mathcal{O}(g)$  (cf. Definition 2.1). Then

- (i) it holds that  $\mathcal{S}(\mathbb{I}_d \bullet g) = \max\{1, \mathcal{S}(g)\},\$
- (*ii*) it holds that  $\mathbb{S}_0(\mathbb{I}_d \bullet g) = \max\{1, \mathbb{S}_0(g)\},\$
- (iii) it holds that  $\mathcal{S}(\not e \bullet \mathbb{I}_d) = \max\{1, \mathcal{S}(\not e)\},\$
- (iv) it holds that  $\mathbb{S}_1(\not e \bullet \mathbb{I}_d) = \max\{1, \mathbb{S}_1(\not e)\},\$
- (v) it holds that  $\mathbb{S}_0(\not f \bullet \mathbb{I}_d) = \mathbb{S}_1(\mathbb{I}_d \bullet g) = 1$ , and
- (vi) it holds that  $\mathcal{S}(\not f \bullet \mathbb{I}_d \bullet g) = \max{\mathcal{S}(\not f), \mathcal{S}(g)}$
- (cf. Definitions 2.6, 2.8, and 2.13).

Proof of Proposition 2.17. Throughout this proof let  $A \in \mathbb{R}^{2d \times d}$  satisfy

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$
 (2.21)

Observe that (2.21) demonstrates that

$$\|A\mathcal{W}_{\mathcal{L}(g),g}\|_{\infty} = \|\mathcal{W}_{\mathcal{L}(g),g}\|_{\infty} \quad \text{and} \quad \|A\mathcal{B}_{\mathcal{L}(g),g} + \mathcal{B}_{1,\mathbb{I}_d}\|_{\infty} = \|\mathcal{B}_{\mathcal{L}(g),g}\|_{\infty} \quad (2.22)$$

(cf. Definitions 2.6 and 2.12). Furthermore, note that (2.9) and (2.21) show that

$$\mathbb{I}_d = \left( \left( A, 0 \right), \left( A^*, 0 \right) \right) \in \left( \left( \mathbb{R}^{2d \times d} \times \mathbb{R}^{2d} \right) \times \left( \mathbb{R}^{d \times 2d} \times \mathbb{R}^d \right) \right).$$
(2.23)

Combining this and (2.22) with (2.10) establishes items (i) and (ii). Observe that Proposition 2.7 and (2.21) imply that

$$\|\mathcal{W}_{1,\ell}(A^*)\|_{\infty} = \|\mathcal{W}_{1,\ell}\|_{\infty}.$$
 (2.24)

Combining this, (2.10), and (2.23) with the fact that  $\|\mathcal{W}_{1,\ell}\mathcal{B}_{2,\mathbb{I}_d} + \mathcal{B}_{1,\ell}\|_{\infty} = \|\mathcal{B}_{1,\ell}\|_{\infty}$  establishes items (iii) and (iv). Note that Lemma 2.16, Corollary 2.15, (2.21), and (2.23) show that

$$\mathbb{S}_0(\not \bullet \mathbb{I}_d) = \mathbb{S}_0(\mathbb{I}_d) = 1 = \mathbb{S}_1(\mathbb{I}_d) = \mathbb{S}_1(\mathbb{I}_d \bullet g).$$
(2.25)

This establishes item (v). Observe that (2.22), (2.23), (2.24), and (2.10) imply that

$$\mathcal{S}(\not \bullet \mathbb{I}_d \bullet g) = \max\{\mathcal{S}(\not e), \mathcal{S}(g)\}.$$
(2.26)

This establishes item (vi). The proof of Proposition 2.17 is thus complete.

**Proposition 2.18** (Sizes of ANN parameters of iterated compositions). Let  $d \in \mathbb{N}$ ,  $a_1, a_2, \ldots, a_d \in \mathbb{N}$ ,  $\aleph_0, \aleph_1, \ldots, \aleph_d \in \mathbb{N}$  satisfy for all  $i \in \{1, 2, \ldots, d\}$  that  $\mathcal{I}(\aleph_i) = a_i = \mathcal{O}(\aleph_{i-1})$  and let  $f \in \mathbb{N}$  satisfy

$$\boldsymbol{\ell} = \boldsymbol{\hbar}_d \bullet \mathbb{I}_{a_d} \bullet \boldsymbol{\hbar}_{d-1} \bullet \mathbb{I}_{a_{d-1}} \bullet \dots \bullet \boldsymbol{\hbar}_1 \bullet \mathbb{I}_{a_1} \bullet \boldsymbol{\hbar}_0 \tag{2.27}$$

(cf. Definitions 2.1, 2.6, and 2.8). Then  $\mathcal{S}(\not{e}) \leq \max\{\mathcal{S}(\aleph_0), \mathcal{S}(\aleph_1), \dots, \mathcal{S}(\aleph_d)\}$  (cf. Definition 2.13).

Proof of Proposition 2.18. Note that Proposition 2.17 and induction ensure that

$$\mathcal{S}(\mathbf{f}) \le \max\{\mathcal{S}(\mathbf{\hbar}_0), \mathcal{S}(\mathbf{\hbar}_1), \dots, \mathcal{S}(\mathbf{\hbar}_d)\}$$
(2.28)

(cf. Definition 2.13). The proof of Proposition 2.18 is thus complete.

## 3 Lower bounds for the minimal number of ANN parameters in the approximation of certain high-dimensional functions

In this section we establish in Corollary 3.4, Proposition 3.21, and Proposition 3.22 below suitable lower bounds for the minimal number of parameters of shallow or insufficiently deep ANNs to approximate certain high-dimensional target functions.

Our proof of Corollary 3.4 uses appropriate lower bounds for the minimal number of ANN parameters to approximate the product functions in Lemma 3.2 and Lemma 3.3. We derrive Lemma 3.2 and Lemma 3.3 from the well known upper bounds for the absolute values of realization functions of ANNs in Lemma 3.1. Lemma 3.1 is a slightly modified variant of, e.g., Grohs et al. [13, Corollary 4.3].

Our proofs of Proposition 3.21 and Proposition 3.22 employ the lower bound result for certain families of oscillating functions in Proposition 3.17. A result similar to Proposition 3.17 can be found, e.g., in Telgarsky [31, Theorem 1.1]. Our proof of Proposition 3.17 uses the essentially well known upper bound result for the number of certain linear regions of realization functions of ANNs in Proposition 3.13. In the scientific literature results related to Proposition 3.13 can be found, e.g., in Raghu et al. [24, Theorem 1]. Our proof of Proposition 3.13, in turn, utilizes the elementary ANN representation result in Lemma 3.12. Lemma 3.12 builds up on the elementary concepts and results regarding intersections of half-spaces in Section 3.2.

#### **3.1** Lower bounds for approximations of product functions

**Lemma 3.1.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $\ell \in \mathbb{N}$  (cf. Definition 2.1). Then it holds for all  $x \in [a, b]^{\mathcal{I}(\ell)}$  that

$$\|(\mathcal{R}(\boldsymbol{\ell}))(\boldsymbol{x})\|_{\infty} \leq \mathcal{I}(\boldsymbol{\ell}) \left[\mathcal{P}(\boldsymbol{\ell}) \max\{\mathcal{S}(\boldsymbol{\ell}), 1\}\right]^{\mathcal{L}(\boldsymbol{\ell})} \max\{|\boldsymbol{a}|, |\boldsymbol{b}|, 1\}$$
(3.1)

(cf. Definitions 2.3, 2.12, and 2.13).

Proof of Lemma 3.1. Throughout this proof assume w.l.o.g. that  $\mathcal{O}(\not) = 1$  and let  $d, L \in \mathbb{N}$ ,  $l_0, l_1, \ldots, l_L \in \mathbb{N}, x_0 \in [a, b]^{\mathcal{I}(\ell)}, x_1 \in \mathbb{R}^{l_1}, x_2 \in \mathbb{R}^{l_2}, \ldots, x_L \in \mathbb{R}^{l_L}$  satisfy for all  $k \in \{1, 2, \ldots, L\}$  that

$$\mathcal{I}(\boldsymbol{f}) = d, \qquad \mathcal{L}(\boldsymbol{f}) = L, \qquad \text{and} \qquad x_k = \Re(\mathcal{W}_{k,\boldsymbol{f}} x_{k-1} + \mathcal{B}_{k,\boldsymbol{f}})$$
(3.2)

(cf. Definition 2.2). Observe that (3.2) shows that for all  $k \in \{1, 2, ..., L\}$  it holds that

$$\|x_{k}\|_{\infty} = \|\Re(\mathcal{W}_{k,\ell}x_{k-1} + \mathcal{B}_{k,\ell})\|_{\infty} \leq l_{k-1}\|\mathcal{W}_{k,\ell}\|_{\infty}\|x_{k-1}\|_{\infty} + \|\mathcal{B}_{k,\ell}\|_{\infty} \\ \leq l_{k-1}\mathcal{S}(\ell)\|x_{k-1}\|_{\infty} + \mathcal{S}(\ell) \\ \leq l_{k-1}\mathcal{S}(\ell)(\|x_{k-1}\|_{\infty} + 1) \\ \leq l_{k-1}\max\{\mathcal{S}(\ell), 1\}2\max\{\|x_{k-1}\|_{\infty}, 1\}.$$
(3.3)

(cf. Definitions 2.12 and 2.13). Combining this and (3.2) with induction demonstrates that

$$\begin{aligned} |(\mathcal{R}(\boldsymbol{\ell}))(x_{0})| &= |\mathcal{W}_{L,\boldsymbol{\ell}}x_{L-1} + \mathcal{W}_{L,\boldsymbol{\ell}}| \leq l_{L-1}\mathcal{S}(\boldsymbol{\ell})2\max\{\|x_{L-1}\|_{\infty}, 1\} \\ &\leq \left(\prod_{k=0}^{L-1}l_{k}\right)\max\{\mathcal{S}(\boldsymbol{\ell}), 1\}^{L}2^{L}\max\{\|x_{0}\|_{\infty}, 1\} \\ &\leq \left(\prod_{k=0}^{L-1}2l_{k}\right)\max\{\mathcal{S}(\boldsymbol{\ell}), 1\}^{L}\max\{|a|, |b|, 1\}. \end{aligned}$$
(3.4)

This, the inequality of arithmetic and geometric means, and the fact that  $l_0 = d$  and  $l_L = 1$  imply that

$$\begin{aligned} |(\mathcal{R}(\boldsymbol{f}))(x_{0})| &\leq \left(\prod_{k=0}^{L-1} 2l_{k}\right) \max\{\mathcal{S}(\boldsymbol{f}), 1\}^{L} \max\{|a|, |b|, 1\} \\ &= d\left(\prod_{k=1}^{L} 2l_{k}\right) \max\{\mathcal{S}(\boldsymbol{f}), 1\}^{L} \max\{|a|, |b|, 1\} \\ &\leq d\left(L^{-1} \sum_{k=1}^{L} 2l_{k}\right)^{L} \max\{\mathcal{S}(\boldsymbol{f}), 1\}^{L} \max\{|a|, |b|, 1\} \\ &\leq d\left(\sum_{k=1}^{L} l_{k}(l_{k-1}+1)\right)^{L} \max\{\mathcal{S}(\boldsymbol{f}), 1\}^{L} \max\{|a|, |b|, 1\} \\ &= d\mathcal{P}(\boldsymbol{f})^{L} \max\{\mathcal{S}(\boldsymbol{f}), 1\}^{L} \max\{|a|, |b|, 1\}. \end{aligned}$$
(3.5)

Hence we obtain (3.1). The proof of Lemma 3.1 is thus complete.

**Lemma 3.2.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $\not \in \mathbb{N}$ ,  $d, L \in \mathbb{N}$ ,  $\varepsilon \in (0, 2^d)$  satisfy

$$\mathcal{L}(\boldsymbol{\ell}) \leq L, \quad \mathcal{R}(\boldsymbol{\ell}) \in C(\mathbb{R}^d, \mathbb{R}), \quad and \quad \sup_{x = (x_1, \dots, x_d) \in [a, b]^d} |(\mathcal{R}(\boldsymbol{\ell}))(x) - \prod_{i=1}^d x_i| \leq \varepsilon \quad (3.6)$$

and  $\max\{|a|, |b|\} \ge 2$  (cf. Definitions 2.1 and 2.3). Then

$$\mathcal{P}(\boldsymbol{\ell}) \max\{\mathcal{S}(\boldsymbol{\ell}), 1\} \ge \left(\frac{2^d - \varepsilon}{2d}\right)^{1/L}.$$
(3.7)

(cf. Definition 2.13).

$$\max\{|a|, |b|\}^{d} = |f(x)| \leq |(\mathcal{R}(\mathcal{F}))(x)| + \varepsilon$$
  

$$\leq d[\mathcal{P}(\mathcal{F}) \max\{\mathcal{S}(\mathcal{F}), 1\}]^{\mathcal{L}(\mathcal{F})} \max\{|a|, |b|, 1\} + \varepsilon$$
  

$$\leq d[\mathcal{P}(\mathcal{F}) \max\{\mathcal{S}(\mathcal{F}), 1\}]^{L} \max\{|a|, |b|\} + \varepsilon$$
  

$$\leq (d[\mathcal{P}(\mathcal{F}) \max\{\mathcal{S}(\mathcal{F}), 1\}]^{L} + \frac{\varepsilon}{2}) \max\{|a|, |b|\}$$
(3.8)

(cf. Definition 2.13). This and the assumption that  $\max\{|a|, |b|\} \ge 2$  imply that

$$2^{d-1} \le \max\{|a|, |b|\}^{d-1} \le d[\mathcal{P}(\mathbf{f}) \max\{\mathcal{S}(\mathbf{f}), 1\}]^L + \frac{\varepsilon}{2}.$$
(3.9)

Hence we obtain (3.7). The proof of Lemma 3.2 is thus complete.

**Lemma 3.3.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $c \in [1, \infty)$ ,  $\varepsilon \in (0, 1]$ ,  $d, L \in \mathbb{N}$  satisfy  $\max\{|a|, |b|\} \ge 2$ , let  $f: [a, b]^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in [a, b]^d$  that  $f(x) = \prod_{i=1}^d x_i$ , and let  $f \in \mathbb{N}$ satisfy  $\mathcal{S}(f) \le cd^c$ ,  $\mathcal{R}(f) \in C(\mathbb{R}^d, \mathbb{R})$ ,  $\mathcal{L}(f) \le L$ , and  $\sup_{x \in [a, b]^d} |(\mathcal{R}(f))(x) - f(x)| \le \varepsilon$  (cf. Definitions 2.1, 2.3, and 2.13). Then

$$\mathcal{P}(\mathbf{f}) \ge 2^{\frac{d-2}{L}} c^{-1} d^{-c-1}. \tag{3.10}$$

Proof of Lemma 3.3. Throughout this proof assume w.l.o.g. that d > 1. Observe that Lemma 3.2 (applied with  $a \land a, b \land b, f \land f, d \land d, L \land L, \varepsilon \land \varepsilon, f \land f$  in the notation of Lemma 3.2) shows that

$$\mathcal{P}(\boldsymbol{\ell}) \ge \left(\frac{2^d - \varepsilon}{2d}\right)^{1/L} \mathcal{S}(\boldsymbol{\ell})^{-1} \ge (2^{d-1} - \frac{1}{2})^{\frac{1}{L}} d^{-\frac{1}{L}} c^{-1} d^{-c} \ge 2^{\frac{d-2}{L}} c^{-1} d^{-c-1}.$$
(3.11)

Hence we obtain (3.10). The proof of Lemma 3.3 is thus complete.

**Corollary 3.4.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $c \in [1, \infty)$ ,  $\varepsilon \in (0, 1]$ ,  $d, L \in \mathbb{N}$  satisfy  $\max\{|a|, |b|\} \ge 2$ and let  $f: [a, b]^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in [a, b]^d$  that  $f(x) = \prod_{i=1}^d x_i$ . Then

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{S}(\not l) \leq cd^c) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{2L}} (4cL)^{-3c} \quad (3.12)$$

Proof of Corollary 3.4. Throughout this proof let  $g: \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that

$$g(x) = 2^{\frac{x}{2L}} x^{-c-1}.$$
(3.13)

Note that (3.13) implies that for all  $x \in \mathbb{R}$  it holds that

$$g'(x) = \ln\left(2^{\frac{1}{2L}}\right)2^{\frac{x}{2L}}x^{-c-1} + 2^{\frac{x}{2L}}(-c-1)x^{-c-2} = 2^{\frac{x}{2L}}x^{-c-2}\left(\frac{x\ln(2)}{2L} - (c+1)\right).$$
(3.14)

This shows that for all  $x \in \left(-\infty, \frac{2L(c+1)}{\ln(2)}\right), y \in \left(\frac{2L(c+1)}{\ln(2)}, \infty\right)$  it holds that

$$g'(x) < 0, \qquad g'\left(\frac{2L(c+1)}{\ln(2)}\right) = 0, \qquad \text{and} \qquad g'(y) > 0.$$
 (3.15)

Combining this and (3.13) with the fact that  $\frac{2}{e \ln(2)} \leq 2$  ensures that

$$\inf_{x \in \mathbb{R}} g(x) = g\left(\frac{2L(c+1)}{\ln(2)}\right) = 2^{\frac{c+1}{\ln(2)}} \left(\frac{2L(c+1)}{\ln(2)}\right)^{-c-1} = \left(\frac{2L(c+1)}{e\ln(2)}\right)^{-c-1} \ge (2L(c+1))^{-c-1}.$$
 (3.16)

This and (3.13) show that

$$2^{\frac{d-2}{L}}c^{-1}d^{-c-1} = 2^{\frac{d}{2L}}2^{\frac{-2}{L}}c^{-1}2^{\frac{d}{2L}}d^{-c-1} = 2^{\frac{d}{2L}}2^{\frac{-2}{L}}c^{-1}g(d) \ge 2^{\frac{d}{2L}}(4c)^{-1}g(d)$$
$$\ge 2^{\frac{d}{2L}}(4c)^{-1}(2L(c+1))^{-c-1}$$
$$\ge 2^{\frac{d}{2L}}(4cL)^{-c-2}$$
$$> 2^{\frac{d}{2L}}(4cL)^{-3c}.$$
(3.17)

Observe that Lemma 3.3 (applied with  $a \curvearrowleft a, b \curvearrowleft b, c \curvearrowleft c, \varepsilon \curvearrowleft \varepsilon, d \curvearrowleft d, L \curvearrowleft L, f \curvearrowleft f$  in the notation of Lemma 3.3) hence demonstrates that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{S}(\not l) \leq cd^c) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d-2}{L}} c^{-1} d^{-c-1}$$
(3.18)  
$$\geq 2^{\frac{d}{2L}} (4cL)^{-3c}.$$

This establishes (3.12). The proof of Corollary 3.4 is thus complete.

#### **3.2** Intersections of half-spaces

**Definition 3.5** (Spaces of affine linear functions). Let  $d \in \mathbb{N}$  and let  $D \subseteq \mathbb{R}^d$  be a non-empty set. Then we denote by  $\mathfrak{L}(D)$  the set given by

$$\mathfrak{L}(D) = \left\{ f \in C(D, \mathbb{R}) : \begin{bmatrix} \exists a_0, a_1, \dots, a_d \in \mathbb{R} \\ \forall x = (x_1, x_2, \dots, x_d) \in D : \\ f(x) = a_0 + \sum_{j=1}^d a_j x_j \end{bmatrix} \right\}.$$
(3.19)

**Lemma 3.6.** Let  $\not{\ell} \in \mathbb{N}$  satisfy  $\mathcal{L}(\not{\ell}) = 1$  (cf. Definition 2.1). Then it holds that  $\mathcal{R}(\not{\ell}) \in \mathfrak{L}(\mathbb{R}^{\mathcal{I}(\not{\ell})})$  (cf. Definitions 2.3 and 3.5).

Proof of Lemma 3.6. Note that (2.5) and (3.19) show that  $\mathcal{R}(\mathcal{L}) \in \mathfrak{L}(\mathbb{R}^{\mathcal{I}(\mathcal{L})})$ . The proof of Lemma 3.6 is thus complete.

**Definition 3.7** (Intersections of half-spaces). Let  $d, k \in \mathbb{N}$ ,  $h = (h_1, \ldots, h_k) \in (\mathfrak{L}(\mathbb{R}^d))^k$ ,  $i = (i_1, \ldots, i_k) \in \{0, 1\}^k$  (cf. Definitions 2.3 and 3.5). Then we denote by  $\operatorname{Hyp}(h, i) \subseteq \mathbb{R}^d$  the set given by

$$Hyp(h,i) = \bigcap_{j=1}^{k} \{ x \in \mathbb{R}^{d} \colon (-1)^{i_{j}} h_{j}(x) \le 0 \}.$$
(3.20)

**Corollary 3.8.** Let  $d, k \in \mathbb{N}$ ,  $h = (h_1, \ldots, h_k) \in (\mathfrak{L}(\mathbb{R}^d))^k$ ,  $i = (i_1, \ldots, i_k) \in \{0, 1\}^k$  and let  $x \in \mathrm{Hyp}(h, i)$  (cf. Definitions 3.5 and 3.7). Then

$$\mathfrak{R}((h_1(x), h_2(x), \dots, h_k(x))) = (i_1 h_1(x), i_2 h_2(x), \dots, i_k h_k(x))$$
(3.21)

(cf. Definition 2.2).

Proof of Corollary 3.8. Observe that (3.20) shows that for all  $j \in \{1, 2, ..., k\}$  it holds that

$$\Re(h_j(x)) = i_j(h_j(x)) \tag{3.22}$$

(cf. Definition 2.2). Hence we obtain (3.21). The proof of Corollary 3.8 is thus complete.

**Corollary 3.9.** Let  $d, k \in \mathbb{N}$ ,  $h = (h_1, \ldots, h_k) \in (\mathfrak{L}(\mathbb{R}^d))^k$ ,  $v \in \mathbb{R}^d \setminus \{0\}$ ,  $w \in \mathbb{R}^d$ ,  $A = \bigcup_{\lambda \in \mathbb{R}} \{w + \lambda v\}$  (cf. Definition 3.5). Then there exist  $i_0, i_1, \ldots, i_k \in \{0, 1\}^k$  such that

$$A \subseteq \left(\bigcup_{j=0}^{k} \operatorname{Hyp}(h, i_{j})\right)$$
(3.23)

(cf. Definition 3.7).

Proof of Corollary 3.9. Throughout this proof assume w.l.o.g. that there exist  $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$  which satisfy for all  $j \in \{1, 2, \ldots, k\}$  that  $\{w + \lambda_j v\} = \{x \in \mathbb{R}^d : h_j(x) = 0\} \cap A$  and  $\lambda_j \leq \lambda_{\min\{j+1,k\}}$ , let  $\lambda_1, \lambda_2, \ldots, \lambda_{k+1} \in (-\infty, \infty]$  satisfy for all  $j \in \{1, 2, \ldots, k\}$  that

$$\lambda_j \le \lambda_{\min\{j+1,k\}} < \lambda_{k+1} = \infty \quad \text{and} \quad \{w + \lambda_j v\} = \{x \in \mathbb{R}^d \colon h_j(x) = 0\} \cap A, \quad (3.24)$$

and let  $B_0, B_1, \ldots, B_k \subseteq A$  satisfy for all  $j \in \{1, 2, \ldots, k\}$  that

$$B_0 = \{ w + \mu v \in \mathbb{R}^n \colon \mu \in (-\infty, \lambda_1) \} \quad \text{and} \quad B_j = \{ w + \mu v \in \mathbb{R}^n \colon \mu \in [\lambda_j, \lambda_{j+1}) \}.$$
(3.25)

Note that (3.24) and the fact that for all  $j \in \{1, 2, ..., k\}$  it holds that  $\{x \in \mathbb{R}^d : h_j(x) = 0\} = \text{Hyp}(h_j, 0) \cap \text{Hyp}(h_j, 1)$  imply that for all  $j \in \{0, 1, ..., k\}$ ,  $j \in \{1, 2, ..., k\}$  there exists  $i \in \{0, 1\}$  such that

$$B_j \subseteq \mathrm{Hyp}(h_{\mathbf{j}}, \mathbf{i})$$
 (3.26)

(cf. Definition 3.7). Hence we obtain that for all  $j \in \{0, 1, ..., k\}$  there exists  $i \in \{0, 1\}^k$  such that

$$B_i \subseteq \mathrm{Hyp}(h, \mathfrak{i}). \tag{3.27}$$

Combining this with the fact that  $A = \bigcup_{j=0}^{k} B_j$  demonstrates (3.23). The proof of Corollary 3.9 is thus complete.

**Definition 3.10** (Convex sets). Let  $d \in \mathbb{N}$  and let  $A \subseteq \mathbb{R}^d$  be a set. Then we denote by  $\mathfrak{C}(A)$  the set given by

$$\mathfrak{C}(A) = \{ C \subseteq A \colon \forall x, y \in C, \lambda \in [0, 1] \colon x + \lambda(y - x) \in C \}.$$
(3.28)

**Corollary 3.11.** Let  $d, k \in \mathbb{N}$  and let  $A \subseteq \mathbb{R}^d$  be a set. Then

- (i) it holds for all  $C_1, C_2, \ldots, C_k \in \mathfrak{C}(A)$  that  $(\bigcap_{i=1}^k C_i) \in \mathfrak{C}(A)$ ,
- (ii) it holds for all  $B \in \mathfrak{C}(A)$ ,  $C \in \mathfrak{C}(\mathbb{R}^d)$  that  $(B \cap C) \in \mathfrak{C}(A)$ , and
- (iii) it holds for all  $h \in \mathfrak{L}(\mathbb{R}^d)$ ,  $i \in \{0, 1\}$  that  $\mathrm{Hyp}(h, i) \in \mathfrak{C}(\mathbb{R}^d)$

(cf. Definitions 3.5, 3.7, and 3.10).

*Proof of Corollary 3.11.* Observe that (3.28) implies items (i) and (ii). Note that (3.20) and (3.28) establish item (iii). The proof of Corollary 3.11 is thus complete.

### 3.3 Lower bounds for approximations of certain classes of oscillating functions

**Lemma 3.12.** Let  $d, L, l_0, l_1, \ldots, l_L \in \mathbb{N}$ ,  $k_0, k_1, \ldots, k_L \in \mathbb{N}_0$  satisfy for all  $s \in \mathbb{N}_0 \cap [0, L]$ that  $L \geq 2$ ,  $l_0 = d$ ,  $l_L = 1$ ,  $k_s = \sum_{j=1}^s l_j$ , let  $f \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})) \subseteq \mathbb{N}$ , for every  $j \in \{1, 2, \ldots, L\}$  let  $\mathfrak{p}_j: (\cup_{s=j}^L \mathbb{R}^{k_s}) \to \mathbb{R}^{k_j}$  satisfy for all  $s \in \mathbb{N} \cap [j, L]$ ,  $i = (i_1, \ldots, i_{k_s}) \in \mathbb{R}^{k_s}$ that  $\mathfrak{p}_j(i) = (i_1, i_2, \ldots, i_{k_j})$ , for every  $k \in \{1, 2, \ldots, L\}$  let

$$(W_{\ell,i,j})_{(i,j)\in\{1,2,\dots,l_{\ell}\}\times\{1,2,\dots,l_{\ell-1}\}} = \mathcal{W}_{k,\ell} \qquad and \qquad (B_{\ell,i})_{i\in\{1,2,\dots,l_{\ell}\}} = \mathcal{B}_{k,\ell}, \tag{3.29}$$

and let  $G_s^i = (g_1^i, \ldots, g_{k_s}^i) \in (\mathfrak{L}(\mathbb{R}^d))^{k_s}$ ,  $s \in \{1, 2, \ldots, L\}$ ,  $i \in \{0, 1\}^{k_{L-1}}$ , satisfy for all  $i \in \{0, 1\}^{k_{L-1}}$ ,  $j \in \{1, 2, \ldots, l_1\}$ ,  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$g_j^i(x) = B_{1,j} + \sum_{p=1}^d W_{1,j,p} x_p$$
(3.30)

and assume for all  $i = (i_1, \ldots, i_{k_{L-1}}) \in \{0, 1\}^{k_{L-1}}, s \in \{1, 2, \ldots, L-1\}, j \in \{1, 2, \ldots, l_{s+1}\}, x \in \mathbb{R}^d$  that

$$g_{k_{s+j}}^{i}(x) = B_{s+1,j} + \sum_{p=1}^{\iota_{s}} W_{s+1,j,p} i_{k_{s-1}+p} \left( g_{k_{s-1}+p}^{i}(x) \right)$$
(3.31)

(cf. Definitions 2.1 and 3.5). Then

- (i) it holds for all  $i, j \in \{0, 1\}^{k_{L-1}}$  that  $G_1^i = G_1^j$ ,
- (*ii*) it holds for all  $s \in \{1, 2, ..., L-1\}$ ,  $i, j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_s(i) = \mathfrak{p}_s(j)$  that

$$G_{s+1}^i = G_{s+1}^j, (3.32)$$

(iii) it holds for all  $x \in \mathbb{R}^d$  that there exists  $i \in \{0, 1\}^{k_{L-1}}$  such that for all  $j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_{L-2}(i) = \mathfrak{p}_{L-2}(j)$  it holds that

$$x \in \operatorname{Hyp}(G_{L-1}^{j}, i) = \operatorname{Hyp}(G_{L-1}^{i}, i), \qquad (3.33)$$

and

(iv) it holds for all  $i \in \{0, 1\}^{k_{L-1}}$  that

$$\mathcal{R}(\boldsymbol{\ell})|_{\mathrm{Hyp}(G_{L-1}^{i},i)} = g_{k_{L}}^{i}|_{\mathrm{Hyp}(G_{L-1}^{i},i)} \in \mathfrak{L}\big(\mathrm{Hyp}\big(G_{L-1}^{i},i\big)\big)$$
(3.34)

(cf. Definitions 2.3 and 3.7).

Proof of Lemma 3.12. Throughout this proof let  $x \in \mathbb{R}^d$ . Observe that (3.30) and the assumption that  $l_1 = k_1$  ensure that for all  $i, j \in \{0, 1\}^{k_{L-1}}$  it holds that

$$G_1^i = G_1^j. (3.35)$$

This establishes item (i). Combining (3.31) and (3.35) ensures that for all  $i, j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_1(i) = \mathfrak{p}_1(j)$  it holds that

$$G_2^i = G_2^j. (3.36)$$

Furthermore, note that (3.31) implies that for all  $s \in \mathbb{N} \cap (0, L-1)$  with  $\forall i, j \in \{0, 1\}^{k_{L-1}}$ :  $[\mathfrak{p}_s(i) = \mathfrak{p}_s(j)] \Rightarrow [G_{s+1}^i = G_{s+1}^j]$  it holds that for all  $i, j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_{s+1}(i) = \mathfrak{p}_{s+1}(j)$  it holds that  $G_{s+2}^i = G_{s+2}^j$ . Combining this and (3.36) with induction shows that for all  $s \in \{1, 2, \ldots, L-1\}, i, j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_s(i) = \mathfrak{p}_s(j)$  it holds that

$$G_{s+1}^i = G_{s+1}^j. (3.37)$$

This establishes item (ii). Observe that (3.35) and the fact that  $l_1 = k_1$  ensure that there exists  $i \in \{0, 1\}^{l_1}$  such that for all  $j \in \{0, 1\}^{k_{L-1}}$  it holds that

$$x \in \mathrm{Hyp}(G_1^j, i) \tag{3.38}$$

(cf. Definition 3.7). This, (3.36), and the fact that  $l_1 + l_2 = k_1 + l_2 = k_2$  demonstrate that there exists  $i \in \{0, 1\}^{k_2}$  such that for all  $j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_1(i) = \mathfrak{p}_1(j)$  it holds that

$$x \in \operatorname{Hyp}(G_2^j, i). \tag{3.39}$$

Moreover, note that (3.31) and (3.37) imply that for all  $s \in \mathbb{N} \cap (1, L-1)$ ,  $i \in \{0, 1\}^{k_s}$  with  $\forall j \in \{0, 1\}^{k_{L-1}}$ :  $[\mathfrak{p}_{s-1}(i) = \mathfrak{p}_{s-1}(j)] \Rightarrow [x \in \operatorname{Hyp}(G_s^j, i)]$  there exists  $\mathfrak{i} \in \{0, 1\}^{k_{s+1}}$  with  $\mathfrak{p}_s(\mathfrak{i}) = i$  such that for all  $j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_s(\mathfrak{i}) = i = \mathfrak{p}_s(j)$  it holds that

$$x \in \mathrm{Hyp}\big(G_{s+1}^{j}, \mathfrak{i}\big). \tag{3.40}$$

Combining this, (3.37), (3.38), and (3.39) with (3.30) and induction ensures that there exists  $i \in \{0, 1\}^{k_{L-1}}$  such that for all  $j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_{L-2}(i) = \mathfrak{p}_{L-2}(j)$  it holds that

$$x \in \mathrm{Hyp}(G_{L-1}^{j}, i) = \mathrm{Hyp}(G_{L-1}^{i}, i).$$
 (3.41)

This establishes item (iii). Next, let  $i = (i_1, \ldots, i_{k_{L-1}}) \in \{0, 1\}^{k_{L-1}}$  and  $y_s \in \mathbb{R}^{l_s}$ ,  $s \in \{0, 1, \ldots, L\}$ , satisfy for all  $s \in \{1, 2, \ldots, L\}$  that

$$y_0 \in \operatorname{Hyp}(G_{L-1}^i, i)$$
 and  $y_s = \mathfrak{R}(\mathcal{W}_{s,\ell}y_{s-1} + \mathcal{B}_{s,\ell})$  (3.42)

(cf. Definition 2.2). Observe that (3.29), (3.30), (3.42), and Corollary 3.8 imply that

$$y_{1} = \Re(\mathcal{W}_{1,\ell}y_{0} + \mathcal{B}_{1,\ell}) = \Re(g_{1}^{i}(y_{0}), g_{2}^{i}(y_{0}), \dots, g_{l_{1}}^{i}(y_{0})) \\ = (i_{1}(g_{1}^{i}(y_{0})), i_{2}(g_{2}^{i}(y_{0})), \dots, i_{l_{1}}(g_{l_{1}}^{i}(y_{0}))).$$
(3.43)

In addition, note that (3.31), (3.42), and Corollary 3.8 demonstrate that for all  $s \in \{1, 2, ..., L-2\}$  with  $y_s = (i_{k_{s-1}+1}(g_{k_{s-1}+1}^i(y_0)), i_{k_{s-1}+2}(g_{k_{s-1}+2}^i(y_0)), ..., i_{k_s}(g_{k_s}^i(y_0)))$  it holds that

$$y_{s+1} = \Re(\mathcal{W}_{s+1,\ell}y_s + \mathcal{B}_{s+1,\ell}) = \Re(g_{k_s+1}^i(y_0), g_{k_s+2}^i(y_0), \dots, g_{k_{s+1}}^i(y_0)) \\ = (i_{k_s+1}(g_{k_s+1}^i(y_0)), i_{k_s+2}(g_{k_s+2}^i(y_0)), \dots, i_{k_{s+1}}(g_{k_{s+1}}^i(y_0))).$$
(3.44)

Combining this and (3.43) with induction ensures that

$$y_{L-1} = (i_{k_{L-2}+1}(g^i_{k_{L-2}+1}(y_0)), i_{k_{L-2}+2}(g^i_{k_{L-2}+2}(y_0)), \dots, i_{k_{L-1}}(g^i_{k_{L-1}}(y_0))).$$
(3.45)

This, (3.31), (3.42), and the fact that  $(\mathcal{R}(\not))(y_0) = W_L y_{L-1} + B_L$  imply that

$$g_{k_{L}}^{i}(y_{0}) = g_{k_{L-1}+1}^{i}(y_{0}) = B_{L,1} + \sum_{p=1}^{l_{L-1}} W_{L,1,p} i_{k_{L-2}+p} \left( g_{k_{L-2}+p}^{i}(y_{0}) \right)$$
  
$$= \mathcal{W}_{L,\ell} y_{L-1} + \mathcal{B}_{L,\ell} = (\mathcal{R}(\ell))(y_{0}).$$
(3.46)

This establishes item (iv). The proof of Lemma 3.12 is thus complete.

**Proposition 3.13.** Let  $a \in [-\infty, \infty)$ ,  $b \in [a, \infty]$ ,  $d \in \mathbb{N}$ ,  $\ell \in \mathbb{N}$ ,  $\mu \in \mathbb{R}^d$ ,  $\nu \in \mathbb{R}^d \setminus \{0\}$  satisfy  $\mathcal{R}(\ell) \in C(\mathbb{R}^d, \mathbb{R})$  and let  $A = [a, b]^d \cap (\bigcup_{\lambda \in \mathbb{R}} \{\mu + \lambda\nu\})$  (cf. Definitions 2.1 and 2.3). Then

$$\min\left(\left\{k \in \mathbb{N} : \begin{bmatrix} \exists B_1, B_2, \dots, B_k \in \mathfrak{C}(A) : \left[(A = \cup_{j=1}^k B_j) \land \\ (\forall j \in \{1, 2, \dots, k\} : \mathcal{R}(\mathbf{f})|_{B_j} \in \mathfrak{L}(B_j))\right] \end{bmatrix}\right\} \cup \{\infty\}\right)$$
$$\leq \left[\frac{\mathcal{P}(\mathbf{f})}{\max\{1, \mathcal{H}(\mathbf{f})\}}\right]^{\max\{1, \mathcal{H}(\mathbf{f})\}}$$
(3.47)

(cf. Definitions 3.5 and 3.10).

Proof of Proposition 3.13. Throughout this proof assume w.l.o.g. that  $\mathcal{L}(\mathbf{f}) > 1$  (cf. Lemma 3.6), let  $P, L, l_0, l_1, \ldots, l_L \in \mathbb{N}, k_0, k_1, \ldots, k_L \in \mathbb{N}_0$  satisfy for all  $s \in \mathbb{N}_0 \cap [0, L]$  that  $\mathcal{D}(\mathbf{f}) = (l_0, l_1, \ldots, l_L), k_s = \sum_{j=1}^s l_j$ , and  $P = \prod_{n=1}^{L-1} (l_n+1)$ , for every  $j \in \{1, 2, \ldots, L\}$  let  $\mathfrak{p}_j : (\cup_{s=j}^L \mathbb{R}^{k_s}) \to \mathbb{R}^{k_j}$  satisfy for all  $s \in \mathbb{N} \cap [j, L], i = (i_1, \ldots, i_{k_s}) \in \mathbb{R}^{k_s}$  that  $\mathfrak{p}_j(i) = (i_1, i_2, \ldots, i_{k_j})$ , and let  $G_s^i = (g_1^i, g_2^i, \ldots, g_{k_s}^i) \in (\mathfrak{L}(\mathbb{R}^d))^{k_s}, s \in \{1, 2, \ldots, L\}, i \in \{0, 1\}^{k_{L-1}}$ , satisfy that

- (I) it holds for all  $i, j \in \{0, 1\}^{k_{L-1}}$  that  $G_1^i = G_1^j$ ,
- (II) it holds for all  $s \in \{1, 2, \dots, L-1\}$ ,  $i, j \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_s(i) = \mathfrak{p}_s(j)$  that

$$G_{s+1}^i = G_{s+1}^j, (3.48)$$

and

(III) it holds for all  $i \in \{0, 1\}^{k_{L-1}}$  that

$$\mathcal{R}(\boldsymbol{\ell})|_{\mathrm{Hyp}(G_{L-1}^{i},i)} = g_{k_{L}}^{i}|_{\mathrm{Hyp}(G_{L-1}^{i},i)} \in \mathfrak{L}(\mathrm{Hyp}(G_{L-1}^{i},i))$$
(3.49)

(cf. Definitions 3.5 and 3.7 and Lemma 3.12). Observe that item (I) and Corollary 3.9 (applied with  $d \curvearrowleft d, k \curvearrowleft l_1, h \curvearrowleft G_1^i, v \curvearrowleft \nu, w \curvearrowleft \mu$  for  $i \in \{0, 1\}^{k_{L-1}}$  in the notation of Corollary 3.9) ensure that there exist  $j_m = (j_{m,1}, \ldots, j_{m,l_1}) \in \{0, 1\}^{l_1}, m \in \mathbb{N}_0 \cap [0, l_1]$ , such that for all  $i \in \{0, 1\}^{k_{L-1}}$  it holds that

$$A \subseteq \left(\bigcup_{m=0}^{l_1} \operatorname{Hyp}(G_1^i, j_m)\right) = \left(\bigcup_{m=0}^{l_1} \left(\bigcap_{p=1}^{l_1} \operatorname{Hyp}(g_p^i, j_{m,p})\right)\right).$$
(3.50)

Furthermore, note that item (II) and Corollary 3.9 (applied with  $d \curvearrowleft d, k \curvearrowleft l_{s+1}, (h_1, h_2, \ldots, h_k) \curvearrowleft (g_{k_s+1}^i, g_{k_s+2}^i, \ldots, g_{k_{s+1}}^i), v \backsim \nu, w \backsim \mu$  for  $s \in \{1, 2, \ldots, L-2\}, i \in \{0, 1\}^{k_{L-1}}$  in the notation of Corollary 3.9) show that for all  $s \in \{1, 2, \ldots, L-2\}, j \in \{0, 1\}^{k_s}$  there exist  $\mathfrak{j}_m = (\mathfrak{j}_{m,1}, \ldots, \mathfrak{j}_{m,l_{s+1}}) \in \{0, 1\}^{l_{s+1}}, m \in \mathbb{N}_0 \cap [0, l_{s+1}],$  such that for all  $i \in \{0, 1\}^{k_{L-1}}$  with  $\mathfrak{p}_s(i) = j$  it holds that

$$A \subseteq \left( \bigcup_{m=0}^{l_{s+1}} \left( \bigcap_{p=1}^{l_{s+1}} \operatorname{Hyp}\left(g_{k_s+p}^i, \mathfrak{j}_{m,p}\right) \right) \right).$$

$$(3.51)$$

Combining this, (3.50), and the assumption that for all  $s \in \{1, 2, ..., L\}$  it holds that  $k_s = \sum_{j=1}^{s} l_j$  and  $P = \prod_{n=1}^{L-1} (l_n+1)$  with induction implies that there exist  $j_m = (j_{m,1}, ..., j_{m,k_{L-1}}) \in \{0,1\}^{k_{L-1}}, m \in \{1, 2, ..., P\}$ , such that

$$A = \bigcup_{m=1}^{P} \left( A \cap \operatorname{Hyp}(G_{L-1}^{j_m}, j_m) \right) = \bigcup_{m=1}^{P} \left( A \cap \left( \bigcap_{p=1}^{k_{L-1}} \operatorname{Hyp}(g_p^{j_m}, j_{m,p}) \right) \right).$$
(3.52)

This, item (III), Corollary 3.11, and the fact that  $A \in \mathfrak{C}(A)$  show that

$$\min\left(\left\{k \in \mathbb{N} : \begin{bmatrix} \exists B_1, B_2, \dots, B_k \in \mathfrak{C}(A) : \left[(A = \bigcup_{b=1}^k B_b) \land \\ (\forall b \in \{1, 2, \dots, k\} : \mathcal{R}(\mathscr{I})|_{B_b} \in \mathfrak{L}(B_b)) \end{bmatrix}\right\} \cup \{\infty\}\right) \le P.$$
(3.53)

Moreover, observe that the inequality of arithmetic and geometric means implies that

$$P = \prod_{k=1}^{L-1} (l_k + 1) \le \left[ \frac{\sum_{k=1}^{L-1} (l_k + 1)}{L - 1} \right]^{L-1} \le \left[ \frac{\sum_{k=1}^{L} l_k (l_{k-1} + 1)}{L - 1} \right]^{L-1} = \left[ \frac{\mathcal{P}(\boldsymbol{\ell})}{L - 1} \right]^{L-1}.$$
 (3.54)

This and (3.53) establish (3.47). The proof of Proposition 3.13 is thus complete.

**Definition 3.14** (Euclidean norm). We denote by  $\|\cdot\|_2 \colon (\bigcup_{d\in\mathbb{N}}\mathbb{R}^d) \to \mathbb{R}$  the function which satisfies for all  $d\in\mathbb{N}$ ,  $x = (x_1,\ldots,x_d)\in\mathbb{R}^d$  that  $\|x\|_2 = \left[\sum_{j=1}^d |x_j|^2\right]^{1/2}$ .

**Lemma 3.15.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d \in \mathbb{N} \cap [3, \infty)$ ,  $\kappa \in (0, \infty)$ , let  $(v_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^d$  satisfy for all  $k \in \mathbb{N}$  that  $v_{k+1} - v_k = v_2 - v_1$ , let  $A = [a, b]^d \cap (\bigcup_{\lambda \in \mathbb{R}} \{\lambda v_1 + (1 - \lambda)v_2\})$ , assume  $\{v_1, v_{2^{d+1}+1}\} \subseteq A$ , and let  $f : \mathbb{R}^d \to \mathbb{R}$  and  $g : \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x \in [a, b]^d$ ,  $k \in \mathbb{N} \cap (1, 2^{d+1}]$ that  $f(v_k) - f(v_{k-1}) = f(v_k) - f(v_{k+1}) \in \{-2\kappa, 2\kappa\}$  and  $|f(x) - g(x)| < \kappa$ . Then

$$\min\left(\left\{k \in \mathbb{N} : \begin{bmatrix} \exists B_1, B_2, \dots, B_k \in \mathfrak{C}(A) : \left[(A = \bigcup_{i=1}^k B_i) \land \\ (\forall i \in \{1, 2, \dots, k\} : g|_{B_i} \in \mathfrak{L}(B_i)) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge 2^d$$
(3.55)

(cf. Definitions 3.5 and 3.10).

Proof of Lemma 3.15. Throughout this proof assume w.l.o.g. that

$$\min\left(\left\{k \in \mathbb{N} : \begin{bmatrix} \exists B_1, B_2, \dots, B_k \in \mathfrak{C}(A) : \left[(A = \bigcup_{i=1}^k B_i) \land \\ (\forall i \in \{1, 2, \dots, k\} : g|_{B_i} \in \mathfrak{L}(B_i)) \end{bmatrix}\right\} \cup \{\infty\}\right) < \infty,$$
(3.56)

let  $N \in \mathbb{N}$ ,  $B_1, B_2, \ldots, B_N \in \mathfrak{C}(A)$  satisfy for all  $j \in \{1, 2, \ldots, N\}$  that  $A = \bigcup_{i=1}^N B_i$  and  $g|_{B_j} \in \mathfrak{L}(B_j)$  (cf. Definitions 3.5 and 3.10). Note that the assumption that for all  $k \in \mathbb{N} \cap (1, 2^{d+1}]$ ,

 $x \in [a,b]^d$  it holds that  $v_k - v_{k-1} = v_{k+1} - v_k$ ,  $f(v_k) - f(v_{k-1}) = f(v_k) - f(v_{k+1}) \in \{-2\kappa, 2\kappa\}$ , and  $|f(x) - g(x)| < \kappa$  implies that for all  $k \in \mathbb{N} \cap (1, 2^{d+1}]$  it holds that

$$\left|g(v_{k}) + \left(\frac{g(v_{k}) - g(v_{k-1})}{\|v_{k} - v_{k-1}\|_{2}}\right)\|v_{k+1} - v_{k}\|_{2} - f(v_{k+1})\right| = |2g(v_{k}) - g(v_{k-1}) - f(v_{k+1})|$$

$$= |2g(v_{k}) - g(v_{k-1}) - f(v_{k-1})|$$

$$> |2f(v_{k}) - f(v_{k-1}) - f(v_{k-1})| - 3\kappa$$

$$= 2|f(v_{k}) - f(v_{k-1})| - 3\kappa$$

$$= 4\kappa - 3\kappa = \kappa$$

$$(3.57)$$

(cf. Definition 3.14). Combining this with the fact that for all  $j \in \{1, 2, ..., N\}$  it holds that  $B_j \in \mathfrak{C}(A)$  and  $g|_{B_j} \in \mathfrak{L}(B_j)$  ensures that for all  $j \in \{1, 2, ..., N\}$ ,  $k \in \mathbb{N} \cap (1, 2^{d+1}]$  with  $v_{k-1}, v_k \in B_j$  it holds that

$$v_{k+1} \notin B_j. \tag{3.58}$$

Furthermore, observe that the fact that for all for all  $j \in \{1, 2, ..., N\}$  it holds that  $B_j \in \mathfrak{C}(A)$ ensures that for all  $j \in \{1, 2, ..., N\}$ ,  $k \in \mathbb{N} \cap (1, 2^{d+1}]$  with  $v_{k-1} \in B_j$ ,  $v_k \notin B_j$  it holds that

$$\nu_{k+1} \notin B_j. \tag{3.59}$$

Combining this and (3.58) with the fact that  $A = \bigcup_{j=1}^{N} B_j$  ensures that for all  $k \in \{1, 2, \dots, 2^d\}$  there exists  $j \in \{1, 2, \dots, N\}$  such that

$$v_{2k-1} \in B_j \qquad \text{and} \qquad v_{2k+1} \notin B_j. \tag{3.60}$$

This, the fact that for all for all  $j \in \{1, 2, ..., N\}$  it holds that  $B_j \in \mathfrak{C}(A)$  ensure that  $N \ge 2^d$ . The proof of Lemma 3.15 is thus complete.

**Proposition 3.16.** For every  $k \in \{1,2\}$  let  $\mathcal{J}_k \in \mathbb{N}$  satisfy  $\mathcal{R}(\mathcal{J}) \in C(\mathbb{R}^k, \mathbb{R})$ , let  $a \in \mathbb{R}$ ,  $b \in [a, \infty), \nu \in \mathbb{R}^2 \setminus \{0\}, \kappa \in (0, \infty), \varepsilon \in [0, \kappa), v_1, v_2, v_3 \in [a, b]^2$  satisfy  $v_3 = v_2 + \nu = v_1 + 2\nu$ , and let  $f \colon \mathbb{R}^2 \to \mathbb{R}$  satisfy for all  $x \in [a, b]^2$  that

$$f(v_2) - f(v_1) = f(v_2) - f(v_3) \in \{-2\kappa, 2\kappa\} \quad and \quad |f(x) - \mathcal{R}(\ell_2)(x)| \le \varepsilon$$
(3.61)

(cf. Definitions 2.1 and 2.3). Then it holds for all  $k \in \{1, 2\}$  that

$$\mathcal{P}(\boldsymbol{\ell}_k) \ge \max\{1, \mathcal{H}(\boldsymbol{\ell}_k)\} 2^{\frac{\kappa}{\max\{1, \mathcal{H}(\boldsymbol{\ell}_k)\}}}.$$
(3.62)

Proof of Proposition 3.16. Note that (2.2) implies that

$$\mathcal{P}(\boldsymbol{\ell}_1) = \sum_{k=1}^{\mathcal{L}(\boldsymbol{\ell}_1)} \mathbb{D}_k(\boldsymbol{\ell}_1) (\mathbb{D}_{k-1}(\boldsymbol{\ell}_1) + 1) \ge 2 \max\{1, \mathcal{H}(\boldsymbol{\ell}_1)\} \ge \max\{1, \mathcal{H}(\boldsymbol{\ell}_1)\} \ge \max\{1, \mathcal{H}(\boldsymbol{\ell}_1)\} \ge 1 \exp\{1, \mathcal{H}(\boldsymbol{\ell}_1)\} = 1 \exp\{1, \mathcal{H}(\boldsymbol$$

Furthermore, observe that the assumption that  $f(v_2) - f(v_1) = f(v_2) - f(v_3) \in \{-2\kappa, 2\kappa\}$ shows that for all  $g \in \mathfrak{L}(\mathbb{R}^2)$  with  $|g(v_1) - f(v_1)| \leq \varepsilon$  and  $|g(v_2) - f(v_2)| \leq \varepsilon$  it holds that

$$|g(v_3) - f(v_3)| = \left| g(v_2) + \left( \frac{g(v_2) - g(v_1)}{\|v_2 - v_1\|_2} \right) \|v_3 - v_2\|_2 - f(v_1) \right|$$
  

$$= |2g(v_2) - g(v_1) - f(v_1)|$$
  

$$\ge |2f(v_2) - f(v_1) - f(v_1)| - 3\varepsilon$$
  

$$= 2|f(v_2) - f(v_1)| - 3\varepsilon$$
  

$$= 4\kappa - 3\varepsilon > \kappa > \varepsilon.$$
(3.64)

Combining this with Lemma 3.6 implies that for all  $g \in \mathbf{N}$  with  $\mathcal{L}(g) = 1$  and  $\mathcal{R}(g) \in C(\mathbb{R}^2, \mathbb{R})$ there exists  $x \in [a, b]^2$  such that

$$|(\mathcal{R}(g))(x) - f(x)| > \varepsilon.$$
(3.65)

Moreover, note that for all  $g \in \mathbf{N}$  with  $\mathcal{R}(g) \in C(\mathbb{R}^2, \mathbb{R})$  and  $\mathcal{L}(g) = 2$  it holds that

$$\mathcal{P}(\boldsymbol{g}) = \sum_{k=1}^{\mathcal{L}(\boldsymbol{g})} \mathbb{D}_k(\boldsymbol{g}) (\mathbb{D}_{k-1}(\boldsymbol{g}) + 1) \ge 4 = \max\{1, \mathcal{H}(\boldsymbol{g})\} 2^{\frac{2}{\max\{1, \mathcal{H}(\boldsymbol{g})\}}}.$$
 (3.66)

In addition, observe that for all  $g \in \mathbf{N}$  with  $\mathcal{R}(g) \in C(\mathbb{R}^2, \mathbb{R})$  and  $\mathcal{L}(g) \geq 3$  it holds that

$$\mathcal{P}(\boldsymbol{g}) = \sum_{k=1}^{\mathcal{L}(\boldsymbol{g})} \mathbb{D}_k(\boldsymbol{g}) (\mathbb{D}_{k-1}(\boldsymbol{g}) + 1) \ge 2 \max\{1, \mathcal{H}(\boldsymbol{g})\} \ge \max\{1, \mathcal{H}(\boldsymbol{g})\} 2^{\frac{2}{\max\{1, \mathcal{H}(\boldsymbol{g})\}}}.$$
 (3.67)

This, (3.65), and (3.66) demonstrate  $\mathcal{P}(\ell_2) \geq \max\{1, \mathcal{H}(\ell_2)\}2^{\frac{2}{\max\{1, \mathcal{H}(\ell_2)\}}}$ . Combining this with (3.63) establishes (3.62). The proof of Proposition 3.16 is thus complete.

**Proposition 3.17.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d, H \in \mathbb{N}$ ,  $\nu \in \mathbb{R}^d \setminus \{0\}$ ,  $\kappa \in (0, \infty)$ ,  $\sigma \in \{-2\kappa, 2\kappa\}$ ,  $\varepsilon \in [0, \kappa)$ , and  $S \colon \mathbb{N} \to \mathbb{N}$  satisfy for all  $n \in \mathbb{N}$  that

$$S(n) = \begin{cases} 1 & : n = 1 \\ 3 & : n = 2 \\ 2^{n+1} + 1 & : n \ge 3, \end{cases}$$
(3.68)

let  $v_k \in [a, b]^d$ ,  $k \in \{1, 2, ..., S(d)\}$ , and  $f \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $k \in \mathbb{N}$  with  $2 \leq k \leq S(d)$ that  $v_k = v_{k-1} + \nu$  and  $f(v_k) - f(v_{k-1}) = \sigma(-1)^k$ , and let  $\ell \in \mathbb{N}$  satisfy for all  $x \in [a, b]^d$  that

$$\mathcal{R}(\mathbf{f}) \in C(\mathbb{R}^d, \mathbb{R}), \quad |f(x) - \mathcal{R}(\mathbf{f})(x)| \le \varepsilon, \quad and \quad H = \max\{1, \mathcal{H}(\mathbf{f})\}$$
(3.69)

(cf. Definitions 2.1 and 2.3). Then  $\mathcal{P}(\mathbf{f}) \geq H2^{\frac{d}{H}}$ .

Proof of Proposition 3.17. Throughout this proof assume w.l.o.g. that  $d \geq 3$  (cf. Proposition 3.16) and let  $A = [a, b]^d \cap (\bigcup_{\lambda \in \mathbb{R}} \{\lambda v_1 + (1 - \lambda) v_2\})$ . Note that (3.69) and Lemma 3.15 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, \nu \curvearrowleft \nu, \kappa \curvearrowleft \kappa, A \curvearrowleft A, (v_1, v_2, \ldots, v_{2^{d+1}+1}) \curvearrowleft (v_1, v_2, \ldots, v_{S(d)}), g \curvearrowleft \mathcal{R}(\mathcal{L}), f \backsim f$  in the notation of Lemma 3.15) ensure that

$$\min\left(\left\{k \in \mathbb{N} : \begin{bmatrix} \exists B_1, B_2, \dots, B_k \in \mathfrak{C}(A) : \left[(A = \bigcup_{i=1}^k B_i) \land \\ (\forall i \in \{1, 2, \dots, k\} : g|_{B_i} \in \mathfrak{L}(B_i)) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge 2^d.$$
(3.70)

Combining this with Proposition 3.13 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, H \curvearrowleft H, \not f \curvearrowleft \not f, A \curvearrowleft A$  in the notation of Proposition 3.13) implies that

$$\left[\frac{\mathcal{P}(\boldsymbol{\ell})}{H}\right]^{H} \geq \min\left(\left\{k \in \mathbb{N} : \begin{bmatrix} \exists B_{1}, B_{2}, \dots, B_{k} \in \mathfrak{C}(A) : \left[(A = \bigcup_{i=1}^{k} B_{i}) \land \right] \\ (\forall i \in \{1, 2, \dots, k\} : g|_{B_{i}} \in \mathfrak{L}(B_{i})) \end{bmatrix}\right\} \cup \{\infty\}\right) \quad (3.71)$$
$$\geq 2^{d}.$$

Hence we obtain that  $\mathcal{P}(\mathbf{\ell}) \geq H2^{\frac{d}{H}}$ . The proof of Proposition 3.17 is thus complete.

#### **3.4** Oscillation properties of certain families of functions

**Corollary 3.18.** Let  $\varphi \in \mathbb{R}$ ,  $\kappa \in (0, \infty)$ ,  $\gamma \in (0, 1]$ ,  $\beta \in [1, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a + 2\pi\gamma^{-1}\beta^{-1}, \infty)$ ,  $g \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x \in \mathbb{R}$  that  $g(x) = \kappa \sin(x + \varphi)$ , let  $f_d \in (\mathbb{R}^d, \mathbb{R})$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d) \in [a, b]^d$  that  $f_d(x) = g(\gamma \beta^d \prod_{i=1}^d x_i)$ , and let  $S \colon \mathbb{N} \to \mathbb{N}$  satisfy for all  $d \in \mathbb{N}$  that

$$S(d) = \begin{cases} 1 & : d = 1 \\ 3 & : d = 2 \\ 2^{d+1} + 1 & : d \ge 3. \end{cases}$$
(3.72)

Then there exist  $(\nu_d, \sigma_d) \in (\mathbb{R}^d \setminus \{0\}) \times \{-2\kappa, 2\kappa\}, d \in \mathbb{N}, and v_{k,d} \in [a, b]^d, k \in \{1, 2, \dots, S(d)\}, d \in \mathbb{N}, such that$ 

- (i) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S(d)]$  that  $v_{k,d} = v_{k-1,d} + \nu_d$ ,
- (ii) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S(d)]$  that  $f_d(v_{k,d}) f_d(v_{k-1,d}) = (-1)^k \sigma_d$ ,
- (iii) it holds for all  $x \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $g(x + 2k\pi) = g(x) \in [-\kappa, \kappa]$ , and
- (iv) it holds for all  $x, y \in \mathbb{R}$  that  $|g(x) g(y)| \le \kappa |x y|$ .

Proof of Corollary 3.18. Throughout this proof let  $c \in [a, a + 6\beta^{-1}] \subseteq [a, b]$  satisfy  $\beta |c| \geq 3$ . Observe that for all  $d \in \mathbb{N}$ ,  $v = (\alpha, c, c, \dots, c) \in [a, b]^d$  it holds that

$$f_d(v) = g\left(\gamma\beta^d \alpha \prod_{i=2}^d c\right) = \kappa \sin(\alpha\gamma\beta^d c^{d-1} + \varphi).$$
(3.73)

Furthermore, note that the fact that for all  $d \in \mathbb{N}$  it holds that  $|(a + \pi \gamma^{-1} \beta^{-d} |c|^{1-d}) \gamma \beta^d c^{d-1} - a \gamma \beta^d c^{d-1}| = \pi$  shows that for all  $d \in \mathbb{N}$  there exists  $\alpha \in [a, a + \pi \gamma^{-1} \beta^{-d} |c|^{1-d})$  such that

$$\kappa |\sin(\alpha \gamma \beta^d c^{d-1} + \varphi)| = \kappa. \tag{3.74}$$

Moreover, observe that for all  $d \in \mathbb{N}$ ,  $\alpha \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  with  $|\sin(\alpha \gamma \beta^d c^{d-1} + \varphi)| = 1$  it holds that

$$\sin((\alpha + k\pi\gamma^{-1}\beta^{-d}|c|^{1-d})\gamma\beta^{d}c^{d-1} + \varphi) = \sin(\alpha\gamma\beta^{d}c^{d-1} + \varphi + k|c|c^{-1}\pi)$$
$$= (-1)^{k}\sin(\alpha\gamma\beta^{d}c^{d-1} + \varphi).$$
(3.75)

In addition, note that the fact that for all  $d \in \mathbb{N} \cap (2, \infty)$  it holds that  $S(d) \leq 3^{d-1}2 \leq 2\beta^{d-1}|c|^{d-1}$  implies that for all  $d \in \mathbb{N}, k \in \{1, 2, \dots, S(d)\}$  it holds that

$$a \le a + k\pi\gamma^{-1}\beta^{-d}|c|^{1-d} \le a + 2\beta^{d-1}|c|^{d-1}\pi\gamma^{-1}\beta^{-d}|c|^{1-d} = a + 2\pi\gamma^{-1}\beta^{-1} \le b.$$
(3.76)

This, (3.73), (3.74), and (3.75) show that there exist  $v_{k,d} \in \mathbb{R}^d$ ,  $k \in \{1, 2, \ldots, S(d)\}$ ,  $d \in \mathbb{N}$ , which satisfy that

- (I) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$  that  $v_{1,1} \in [a, a + \pi \gamma^{-1} \beta^{-d} |c|^{1-d}) \subseteq [a, b]$  and  $v_{1,d} \in [a, a + \pi \gamma^{-1} \beta^{-d} |c|^{1-d}) \times \{c\}^{d-1} \subseteq [a, b]^d$ ,
- (II) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S(d)]$  that  $v_{k,d} = v_{k-1,d} + (\pi \gamma^{-1} \beta^{-d} |c|^{1-d}, 0, 0, \dots, 0) \in [a, b] \times \{c\}^{d-1} \subseteq [a, b]^d$ , and

(III) it holds for all  $d \in \mathbb{N}$  that there exists  $\sigma \in \{-\kappa, \kappa\}$  such that for all  $k \in \{1, 2, \dots, S(d)\}$  it holds that  $f_d(v_{k,d}) = \sigma(-1)^k$ .

Combining item (I), item (II), and item (III) with the fact that for all  $x, y \in \mathbb{R}, k \in \mathbb{Z}$  it holds that  $\sin(x+2k\pi) = \sin(x)$  and  $|\sin(x) - \sin(y)| \le |x-y|$  establishes items (i), (ii), (iii), and (iv). The proof of Corollary 3.18 is thus complete.

**Corollary 3.19.** Let  $\varphi \in \mathbb{R}$ ,  $\gamma, \kappa \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a + \pi \gamma^{-1}, \infty)$ ,  $g \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x \in \mathbb{R}$  that  $g(x) = \kappa \sin(x + \varphi)$ , let  $f_d \in (\mathbb{R}^d, \mathbb{R})$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d) \in [a, b]^d$  that  $f_d(x) = g(\gamma 2^d \sum_{i=1}^d x_i)$ , let  $S \colon \mathbb{N} \to \mathbb{N}$  satisfy for all  $d \in \mathbb{N}$  that

$$S(d) = \begin{cases} 1 & : d = 1 \\ 3 & : d = 2 \\ 2^{d+1} + 1 & : d \ge 3. \end{cases}$$
(3.77)

Then there exist  $(\nu_d, \sigma_d) \in (\mathbb{R}^d \setminus \{0\}) \times \{-2\kappa, 2\kappa\}, d \in \mathbb{N}, and v_{k,d} \in [a, b]^d, k \in \{1, 2, \dots, S(d)\}, d \in \mathbb{N}, such that$ 

- (i) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S(d)]$  that  $v_{k,d} = v_{k-1,d} + \nu_d$ ,
- (ii) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S(d)]$  that  $f_d(v_{k,d}) f_d(v_{k-1,d}) = (-1)^k \sigma_d$ ,
- (iii) it holds for all  $x \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $g(x + 2k\pi) = g(x) \in [-\kappa, \kappa]$ , and
- (iv) it holds for all  $x, y \in \mathbb{R}$  that  $|g(x) g(y)| \le \kappa |x y|$ .

Proof of Corollary 3.19. Observe that for all  $d \in \mathbb{N}$ ,  $v = (\alpha, \alpha, \dots, \alpha) \in [a, b]^d$  it holds that

$$f_d(v) = g\left(\gamma 2^d \sum_{i=1}^d \alpha\right) = \kappa \sin(\gamma 2^d d\alpha + \varphi).$$
(3.78)

Furthermore, note that the fact that for all  $d \in \mathbb{N}$  it holds that  $|\gamma 2^d da - \gamma 2^d d(a + \pi \gamma^{-1} 2^{-d} d^{-1})| = \pi$  shows that for all  $d \in \mathbb{N}$  there exists  $\alpha \in [a, a + \pi \gamma^{-1} 2^{-d} d^{-1})$  such that

$$|\sin(\gamma 2^d d\alpha + \varphi)| = 1. \tag{3.79}$$

Moreover, observe that for all  $d \in \mathbb{N}$ ,  $\alpha \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  with  $|\sin(\gamma 2^d d\alpha + \varphi)| = 1$  it holds that

$$\sin(\gamma 2^d d(\alpha + k\pi\gamma^{-1}2^{-d}d^{-1}) + \varphi) = \sin(\gamma 2^d d\alpha + \varphi + k\pi) = (-1)^k \sin(\gamma 2^d d\alpha + \varphi).$$
(3.80)

In addition, note that the fact that for all  $d \in \mathbb{N}$  it holds that  $S(d) \leq 2^d d$  implies that for all  $d \in \mathbb{N}, k \in \{1, 2, \dots, S(d)\}$  it holds that

$$a \le a + k\pi\gamma^{-1}2^{-d}d^{-1} \le a + \pi\gamma^{-1} \le b.$$
(3.81)

This, (3.78), (3.79), and (3.80) show that there exist  $v_{k,d} \in \mathbb{R}^d$ ,  $k \in \{1, 2, \ldots, S(d)\}$ ,  $d \in \mathbb{N}$ , which satisfy that

(I) it holds for all  $d \in \mathbb{N}$  that  $v_{1,d} \in [a, a + \pi \gamma^{-1} 2^{-d} d^{-1})^d \subseteq [a, b]^d$ ,

(II) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S(d)]$  that

$$v_{k,d} = v_{k-1,d} + (\pi \gamma^{-1} 2^{-d} d^{-1}, \pi \gamma^{-1} 2^{-d} d^{-1}, \dots, \pi \gamma^{-1} 2^{-d} d^{-1}) \in [a, b]^d,$$
(3.82)

and

(III) it holds for all  $d \in \mathbb{N}$  that there exists  $\sigma \in \{-\kappa, \kappa\}$  such that for all  $k \in \{1, 2, \dots, S(d)\}$  it holds that  $f_d(v_{k,d}) = \sigma(-1)^k$ .

Combining items (I), (II), and (III) with the fact that for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  it holds that  $\sin(x + 2k\pi) = \sin(x)$  and  $|\sin(x) - \sin(y)| \le |x - y|$  establishes items (i), (ii), (iii), and (iv). The proof of Corollary 3.19 is thus complete.

## 3.5 Lower bounds for approximations of specific families of oscillating functions

**Lemma 3.20.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \infty)$ , let  $h: \mathbb{N} \to \mathbb{N}$  satisfy for all  $l \in \mathbb{N}$  that  $h(l) = \max\{1, l-1\}$ , and let  $f: \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $L \in \mathbb{N}$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) = L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge h(L)2^{\frac{d}{h(L)}}$$
(3.83)

(cf. Definitions 2.1 and 2.3). Then it holds for all  $L \in \mathbb{N}$  that

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not l \in \mathbf{N} \colon (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \le L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge 2^{\frac{d}{h(L)}}$$
(3.84)

Proof of Lemma 3.20. Observe that (3.83) and the fact that for all  $L \in \mathbb{N}$  it holds that  $h(L) \leq h(L+1)$  show that for all  $L \in \mathbb{N}$  it holds that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists f \in \mathbf{N} : (\mathcal{P}(f) = p) \land (\mathcal{L}(f) \leq L) \land \\ (\mathcal{R}(f) \in C(\mathbb{R}^{d}, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^{d}} | (\mathcal{R}(f))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right)$$

$$= \min_{l \in \{1,2,\dots,L\}} \min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists f \in \mathbf{N} : (\mathcal{P}(f) = p) \land (\mathcal{L}(f) = l) \land \\ (\mathcal{R}(f) \in C(\mathbb{R}^{d}, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^{d}} | (\mathcal{R}(f))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right)$$
(3.85)
$$\geq \min_{l \in \{1,2,\dots,L\}} h(l) 2^{\frac{d}{h(l)}} \geq h(1) 2^{\frac{d}{h(L)}} = 2^{\frac{d}{h(L)}}.$$

The proof of Lemma 3.20 is thus complete.

**Proposition 3.21.** Let  $\varphi \in \mathbb{R}$ ,  $\gamma \in (0,1]$ ,  $\beta \in [1,\infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a + 2\pi\gamma^{-1}\beta^{-1},\infty)$ ,  $\kappa \in (0,\infty)$  and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$  that  $f_d(x) = \kappa \sin(\gamma \beta^d (\prod_{i=1}^d x_i) + \varphi)$ . Then it holds for all  $d \in \mathbb{N}$ ,  $H \in \mathbb{N}_0$ ,  $\varepsilon \in (0,\kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{H}(\not l) \le H) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge 2^{\frac{d}{\max\{1,H\}}}$$
(3.86)

(cf. Definitions 2.1 and 2.3).

Proof of Proposition 3.21. Throughout this proof let  $h: \mathbb{N} \to \mathbb{N}$  satisfy for all  $L \in \mathbb{N}$  that  $h(L) = \max\{1, L-1\}$  and let  $S_d \in \mathbb{N}, d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N} \cap [3, \infty)$  that  $S_1 = 1, S_2 = 3$ , and  $S_d = 2^{d+1} + 1$ . Note that Corollary 3.18 (applied with  $\varphi \curvearrowleft \varphi, \kappa \curvearrowleft \kappa, \gamma \curvearrowleft \gamma, \beta \curvearrowleft \beta, a \curvearrowleft a, b \curvearrowleft b, f_d \curvearrowleft f_d, S(d) \curvearrowleft S_d$  for  $d \in \mathbb{N}$  in the notation of Corollary 3.18) demonstrates that there exist  $(\nu_d, \sigma_d) \in (\mathbb{R}^d \setminus \{0\}) \times \{-2\kappa, 2\kappa\}, d \in \mathbb{N}$ , and  $v_{k,d} \in [a, b]^d, k \in \{1, 2, \ldots, S_d\}, d \in \mathbb{N}$ , such that

- (I) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S_d]$  that  $v_{k,d} = v_{k-1,d} + \nu_d$ ,
- (II) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N} \cap (1, S_d]$  that  $f_d(v_{k,d}) f_d(v_{k-1,d}) = (-1)^k \sigma_d$ ,
- (III) it holds for all  $x \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $g(x + 2k\pi) = g(x) \in [-\kappa, \kappa]$ , and
- (IV) it holds for all  $x, y \in \mathbb{R}$  that  $|g(x) g(y)| \le \kappa |x y|$ .

Observe that Proposition 3.17 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, H \curvearrowleft h(L), L \curvearrowleft L, \nu \curvearrowleft \nu_d, \\ \kappa \curvearrowleft \kappa, \sigma \curvearrowleft \sigma_d, \varepsilon \curvearrowleft \varepsilon, S \curvearrowleft S_d, (v_1, v_2, \ldots, v_{S_d}) \curvearrowleft (v_{1,d}, v_{2,d}, \ldots, v_{S_d,d}), f \curvearrowleft f_d \text{ for } d, L \in \mathbb{N}, \\ \varepsilon \in (0, \kappa) \text{ in the notation of Proposition 3.17) shows that for all } d, L \in \mathbb{N}, \varepsilon \in (0, \kappa) \text{ it holds that}$ 

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) = L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge h(L)2^{\frac{d}{h(L)}}$$
(3.87)

(cf. Definitions 2.1 and 2.3). This and Lemma 3.20 demonstrate that for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  it holds that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not f \in \mathbf{N} : (\mathcal{P}(\not f) = p) \land (\mathcal{L}(\not f) \leq L) \land \\ (\mathcal{R}(\not f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not f))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{h(L)}}.$$
 (3.88)

The proof of Proposition 3.21 is thus complete.

**Proposition 3.22.** Let  $\varphi \in \mathbb{R}$ ,  $\gamma, \kappa \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a + \pi \gamma^{-1}, \infty)$  and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \kappa \sin(\gamma 2^d (\sum_{i=1}^d x_i) + \varphi)$ . Then it holds for all  $d \in \mathbb{N}$ ,  $H \in \mathbb{N}_0$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{H}(\not l) \le H) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge 2^{\frac{d}{\max\{1,H\}}}$$
(3.89)

(cf. Definitions 2.1 and 2.3).

Proof of Proposition 3.22. Throughout this proof let  $h: \mathbb{N} \to \mathbb{N}$  satisfy for all  $L \in \mathbb{N}$  that  $h(L) = \max\{1, L-1\}$  and let  $S_d \in \mathbb{N}, d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N} \cap [3, \infty)$  that  $S_1 = 1, S_2 = 3$ , and  $S_d = 2^{d+1} + 1$ . Note that Corollary 3.19 (applied with  $\varphi \curvearrowleft \varphi, \kappa \curvearrowleft \kappa, \gamma \curvearrowleft \gamma, \beta \curvearrowleft \beta, a \curvearrowleft a, b \curvearrowleft b, f_d \curvearrowleft f_d, S(d) \curvearrowleft S_d$  for  $d \in \mathbb{N}$  in the notation of Corollary 3.19) demonstrates that there exist  $(\nu_d, \sigma_d) \in (\mathbb{R}^d \setminus \{0\}) \times \{-2\kappa, 2\kappa\}, d \in \mathbb{N}$ , and  $v_{k,d} \in [a, b]^d, k \in \{1, 2, \ldots, S_d\}, d \in \mathbb{N}$ , such that

- (I) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N}(1, S_d]$  that  $v_{k,d} = v_{k-1,d} + \nu_d$ ,
- (II) it holds for all  $d \in \mathbb{N} \cap (1, \infty)$ ,  $k \in \mathbb{N}(1, S_d]$  that  $f_d(v_{k,d}) f_d(v_{k-1,d}) = (-1)^k \sigma_d$ ,

(III) it holds for all  $x \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $g(x + 2k\pi) = g(x) \in [-\kappa, \kappa]$ , and

(IV) it holds for all  $x, y \in \mathbb{R}$  that  $|g(x) - g(y)| \le \kappa |x - y|$ .

Observe that Proposition 3.17 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, H \curvearrowleft h(L), L \curvearrowleft L, \nu \curvearrowleft \nu_d, \\ \kappa \curvearrowleft \kappa, \sigma \curvearrowleft \sigma_d, \varepsilon \curvearrowleft \varepsilon, S \curvearrowleft S_d, (v_1, v_2, \ldots, v_{S_d}) \curvearrowleft (v_{1,d}, v_{2,d}, \ldots, v_{S_d,d}), f \curvearrowleft f_d \text{ for } d, L \in \mathbb{N}, \\ \varepsilon \in (0, \kappa) \text{ in the notation of Proposition 3.17) shows that for all } d, L \in \mathbb{N}, \varepsilon \in (0, \kappa) \text{ it holds that}$ 

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix}\exists \not l \in \mathbf{N} \colon (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) = L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge h(L)2^{\frac{d}{h(L)}}$$
(3.90)

(cf. Definitions 2.1 and 2.3). This and Lemma 3.20 demonstrate that for all  $d, L \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  it holds that

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not e \in \mathbf{N} \colon (\mathcal{P}(\not e) = p) \land (\mathcal{L}(\not e) \leq L) \land \\ (\mathcal{R}(\not e) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not e))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{h(L)}}.$$
 (3.91)

The proof of Proposition 3.22 is thus complete.

## 4 Upper bounds for the minimal number of ANN parameters in the approximation of certain high-dimensional functions

In this section we establish in Corollary 4.16, Corollary 4.28, and Corollary 4.31 below suitable upper bounds for the minimal number of parameters of ANNs to approximate the product functions (Corollary 4.16) and certain highly oscillating functions (Corollary 4.28 and Corollary 4.31) in the case where the absolute values of the parameters of the ANNs are assumed to be uniformly bounded by 1.

Our proof of Corollary 4.16 employs the elementary result regarding the reduction of the absolute value of the size of the parameters of an ANN without changing its realization function in Corollary 4.4 and the essentially well known upper bounds for the minimal number of parameters of ANNs to approximate certain scaled product functions in Lemma 4.15. Lemma 4.15 is an extended variant of, e.g., Beneventano et al. [3, Proposition 6.8]. Our proof of Lemma 4.15 utilizes the elementary result regarding suitable deep ANNs whose realization functions agree with appropriate one-dimensional scaling functions in Corollary 4.6 and the essentially well known upper bound result for the minimal number of parameters of ANNs approximating the product functions in Lemma 4.14. Lemma 4.14 is a slightly extended variant of, e.g., Beneventano et al. [3, Lemma 6.7] and our proof of Lemma 4.14 as well as the auxiliary results in Section 4.3 are strongly inspired by the findings in Beneventano et al. [3, Section 6].

Our proof of Corollary 4.28 employs the elementary result regarding the reduction of the absolute value of the size of the parameters of an ANN without changing its realization function in Lemma 4.3 and the upper bounds for the minimal number of parameters of ANNs approximating compositions of certain periodic functions and certain scaled product functions in Lemma 4.27. Our proof of Lemma 4.27, in turn, combines Corollary 4.6 and Lemma 4.15 with the essentially well known upper bound result for the minimal number of parameters of ANNs approximating certain periodic functions in Lemma 4.24. Our proof of Lemma 4.24 employs the essentially well known ANN approximation result for certain one-dimensional Lipschitz continuous functions in Lemma 4.23 and builds up on the essentially well known properties of sawtooth functions (suitable one-dimensional piecewise linear functions with compact support) in Lemma 4.17 and Lemma 4.18. The results in Lemma 4.17 and Lemma 4.18 are extensions of, e.g., Telgarsky [31, Section 2.2] and Lemma 4.23 is inspired by Beneventano et al. [3, Subsection 4.1].

Our proof of Corollary 4.31 employs Lemma 4.3 as well as the upper bounds for the minimal number of parameters of ANNs approximating compositions of certain periodic functions and scaled sum functions in Corollary 4.30. Our proof of Corollary 4.30, in turn, utilizes Corollary 4.6 and Lemma 4.24.

#### 4.1 Trade-off between the number and the size of ANN parameters

**Corollary 4.1.** Let  $\not{\ell} \in \mathbb{N}$ ,  $L \in \mathbb{N}$  satisfy  $\mathcal{O}(\not{\ell}) = 1$  and  $L > \mathcal{L}(\not{\ell})$ . Then there exists  $g \in \mathbb{N}$  which satisfies that

- (i) it holds that  $\mathcal{R}(\mathcal{L}) = \mathcal{R}(\mathcal{Q})$ ,
- (ii) it holds for all  $k \in \mathbb{N}_0 \cap [0, L]$  that

$$\mathbb{D}_{k}(\boldsymbol{g}) = \begin{cases} \mathbb{D}_{k}(\boldsymbol{f}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\boldsymbol{f})) \\ 2 & : k \in \mathbb{N} \cap [\mathcal{L}(\boldsymbol{f}), L) \\ 1 & : k = L, \end{cases}$$

$$(4.1)$$

- (iii) it holds that  $\mathcal{L}(q) = L$ , and
- (iv) it holds that  $\mathbb{S}_0(q) = \max\{1, \mathbb{S}_0(p)\}, \mathbb{S}_1(q) = 1, \text{ and } \mathcal{S}(q) = \max\{1, \mathcal{S}(p)\}$
- (cf. Definitions 2.1, 2.3, and 2.13).

Proof of Corollary 4.1. Throughout this proof let  $\aleph_1, \aleph_2, \ldots, \aleph_L \in \mathbb{N}$  satisfy for all  $k \in \{2, 3, \ldots, L\}$  that

$$\hbar_1 = \mathbb{I}_1 \quad \text{and} \quad \hbar_k = \mathbb{I}_1 \bullet \hbar_{k-1}$$
(4.2)

(cf. Definitions 2.1, 2.6, and 2.8). Combining (4.2), Lemma 2.11, and Proposition 2.7 with induction shows that

$$\mathcal{L}(\mathscr{h}_{L-\mathcal{L}(\mathscr{f})}) = L - \mathcal{L}(\mathscr{f}) + 1 \quad \text{and} \quad \mathcal{D}(\mathscr{h}_{L-\mathcal{L}(\mathscr{f})}) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{L-\mathcal{L}(\mathscr{f})+2}.$$
(4.3)

This, Proposition 2.10, and Lemma 2.11 imply that for all  $k \in \mathbb{N}_0 \cap [0, L]$  it holds that

$$\mathcal{L}(\mathscr{h}_{L-\mathcal{L}(f)} \bullet f) = L \quad \text{and} \quad \mathbb{D}_{k}(\mathscr{h}_{L-\mathcal{L}(f)} \bullet f) = \begin{cases} \mathbb{D}_{k}(f) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(f)) \\ 2 & : k \in \mathbb{N} \cap [\mathcal{L}(f), L) \\ 1 & : k = L. \end{cases}$$
(4.4)

Furthermore, note that Proposition 2.10 and Proposition 2.7 demonstrate that for all  $k \in \mathbb{N} \cap (0, L - \mathcal{L}(\mathbf{f}))$  it holds that

$$(\mathcal{R}(\mathscr{A}_{k+1}))(x) = (\mathcal{R}(\mathbb{I}_1 \bullet \mathscr{A}_k))(x) = (\mathcal{R}(\mathbb{I}_1))((\mathcal{R}(\mathscr{A}_k))(x)) = (\mathcal{R}(\mathscr{A}_k))(x).$$
(4.5)

This, Proposition 2.7, and induction ensure that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathscr{R}_{L-\mathcal{L}(f)}))(x) = (\mathcal{R}(\mathscr{R}_{1}))(x) = (\mathcal{R}(\mathbb{I}_{1}))(x) = x.$$

$$(4.6)$$

Combining this and the assumption that  $\mathcal{O}(\mathbf{p}) = 1$  with Proposition 2.10 implies that

$$(\mathcal{R}(\mathscr{h}_{L-\mathcal{L}(f)} \bullet f))(x) = (\mathcal{R}(\mathscr{h}_{L-\mathcal{L}(f)}))((\mathcal{R}(f))(x)) = (\mathcal{R}(f))(x).$$
(4.7)

Moreover, observe that (4.2), Proposition 2.17, and induction show that  $\mathbb{S}_1(\mathbb{A}_{L-\mathcal{L}(\ell)} \bullet \ell) = 1$ ,

$$\mathbb{S}_{0}(\mathscr{h}_{L-\mathcal{L}(f)} \bullet f) = \max\{1, \mathbb{S}_{0}(f)\}, \quad \text{and} \quad \mathcal{S}(\mathscr{h}_{L-\mathcal{L}(f)} \bullet f) = \max\{1, \mathcal{S}(f)\}.$$
(4.8)

Combining this, (4.4), and (4.7) establishes items (i), (ii), (iii), and (iv). The proof of Corollary 4.1 is thus complete.

**Corollary 4.2.** Let  $\not{\ell} \in \mathbb{N}$ ,  $L, d \in \mathbb{N}$  satisfy  $\mathcal{R}(\not{\ell}) \in C(\mathbb{R}^d, \mathbb{R})$  and  $\mathcal{L}(\not{\ell}) = L$  (cf. Definitions 2.1 and 2.3). Then there exists  $g \in \mathbb{N}$  such that

- (i) it holds for all  $x \in \mathbb{R}^d$  that  $(\mathcal{R}(q))(x) = 2^{-L}(\mathcal{R}(p))(x)$ ,
- (ii) it holds that  $\mathcal{L}(q) = L$
- (iii) it holds that  $\mathcal{S}(q) \leq 2^{-1} \mathcal{S}(\mathbf{f})$ , and
- (iv) it holds that  $\mathcal{D}(q) = \mathcal{D}(\boldsymbol{\ell})$
- (cf. Definition 2.13).

Proof of Corollary 4.2. Throughout this proof let  $q \in \mathbf{N}$  satisfy for all  $k \in \{1, 2, \dots, L\}$  that

$$\mathcal{L}(\boldsymbol{g}) = L, \qquad \mathcal{W}_{k,\boldsymbol{g}} = 2^{-1} \mathcal{W}_{k,\boldsymbol{\ell}}, \qquad \text{and} \qquad \mathcal{B}_{k,\boldsymbol{g}} = 2^{-k} \mathcal{B}_{k,\boldsymbol{\ell}},$$
(4.9)

and let  $x_0, y_0 \in \mathbb{R}^{l_0}, x_1, y_1 \in \mathbb{R}^{l_1}, \ldots, x_L, y_L \in \mathbb{R}^{l_L}$  satisfy for all  $k \in \{1, 2, \ldots, L\}$  that

$$x_0 = y_0, \qquad x_k = \Re(\mathcal{W}_{k,\ell} x_{k-1} + \mathcal{B}_{k,\ell}), \qquad \text{and} \qquad y_k = \Re(\mathcal{W}_{k,g} y_{k-1} + \mathcal{B}_{k,g}). \tag{4.10}$$

Note that (4.9) implies that

$$\mathcal{S}(q) \le 2^{-1} \mathcal{S}(\ell)$$
 and  $\mathcal{D}(q) = \mathcal{D}(\ell).$  (4.11)

Furthermore, observe that (4.10) demonstrates that for all  $k \in \mathbb{N} \cap (0, L)$  with  $y_k = 2^{-k} x_k$  it holds that

$$y_{k+1} = \mathfrak{R}(\mathcal{W}_{k+1,\mathscr{G}}y_k + \mathcal{B}_{k+1,\mathscr{G}}) = \mathfrak{R}(\mathcal{W}_{k+1,\mathscr{G}}(2^{-k}x_k) + \mathcal{B}_{k+1,\mathscr{G}})$$
$$= \mathfrak{R}(2^{-(k+1)}(\mathcal{W}_{k+1,\mathscr{G}}x_k + \mathcal{B}_{k+1,\mathscr{G}}))$$
$$= 2^{-(k+1)}x_{k+1}.$$
(4.12)

Combining this and (4.10) with induction shows that  $y_{L-1} = 2^{-(L-1)}x_{L-1}$ . Hence (2.5) and (4.10) imply that

$$(\mathcal{R}(\boldsymbol{g}))(x_0) = \mathcal{W}_{L,\boldsymbol{g}} y_{L-1} + \mathcal{B}_{L,\boldsymbol{g}} = \mathcal{W}_{L,\boldsymbol{g}} (2^{-(L-1)} x_{L-1}) + \mathcal{B}_{L,\boldsymbol{g}}$$
$$= 2^{-L} (\mathcal{W}_{L,\boldsymbol{\ell}} x_{L-1} + \mathcal{B}_{L,\boldsymbol{\ell}})$$
$$= 2^{-L} (\mathcal{R}(\boldsymbol{\ell}))(x_0).$$
(4.13)

This and (4.11) establish items (i), (ii), (iii), and (iv). The proof of Corollary 4.2 is thus complete.

**Lemma 4.3.** Let  $\mathcal{f} \in \mathbb{N}$ ,  $d \in \mathbb{N}$  satisfy  $\mathcal{R}(\mathcal{f}) \in C(\mathbb{R}^d, \mathbb{R})$  (cf. Definitions 2.1 and 2.3). Then there exists  $\mathcal{g} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(q) = \mathcal{R}(p)$ ,
- (ii) it holds that  $\mathcal{L}(q) = 2\mathcal{L}(\mathbf{f}) + 1$ ,
- (iii) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g)]$  that

$$\mathbb{D}_{k}(\boldsymbol{g}) = \begin{cases} \mathbb{D}_{k}(\boldsymbol{f}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\boldsymbol{f})) \\ 2 & : k = \mathcal{L}(\boldsymbol{f}) \\ 4 & : k \in \mathbb{N} \cap (\mathcal{L}(\boldsymbol{f}), \mathcal{L}(\boldsymbol{g})) \\ 1 & : k = \mathcal{L}(\boldsymbol{g}), \end{cases}$$
(4.14)

- (iv) it holds that  $\mathcal{P}(q) \leq \mathcal{P}(\boldsymbol{\ell}) + \mathbb{D}_{\mathcal{H}(\boldsymbol{\ell})}(\boldsymbol{\ell}) + 20\mathcal{L}(\boldsymbol{\ell}) \leq 2\mathcal{P}(\boldsymbol{\ell}) + 20\mathcal{L}(\boldsymbol{\ell})$ , and
- (v) it holds that  $\mathcal{S}(q) \leq \max\{1, 2^{-1}\mathcal{S}(p)\}$
- (cf. Definition 2.13).

Proof of Lemma 4.3. Throughout this proof let  $L \in \mathbb{N}$  satisfy  $L = \mathcal{L}(\mathcal{P})$ . Note that Corollary 4.2 (applied with  $\mathcal{P} \curvearrowright \mathcal{P}, L \curvearrowleft L, d \curvearrowleft d$  in the notation of Corollary 4.2) shows that there exists  $\mathcal{Q}_1 \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}^d$  that  $(\mathcal{R}(g_1))(x) = 2^{-L}(\mathcal{R}(f))(x)$ ,
- (II) it holds that  $\mathcal{L}(q_1) = L$
- (III) it holds that  $\mathcal{S}(g_1) \leq 2^{-1} \mathcal{S}(f)$ , and
- (IV) it holds that  $\mathcal{D}(g_1) = \mathcal{D}(f)$

(cf. Definition 2.13). Observe that Lemma 4.5 (applied with  $\beta \curvearrowleft 1$ ,  $B \curvearrowleft 2$ ,  $n \curvearrowleft L$  in the notation of Lemma 4.5) shows that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_2))(x) = 2^L x$ ,
- (B) it holds that  $\mathcal{D}(g_2) = (1, 4, 4, \dots, 4, 1) \in \mathbb{N}^{L+2}$ ,
- (C) it holds that  $\mathcal{S}(g_2) = 1$ , and
- (D) it holds that  $\mathcal{P}(g_2) = (4L 4)2^2 + (2L + 4)2 + 1 = 20L 7 \le 20L.$

Note that item (I), item (A), Proposition 2.10, and Proposition 2.7 imply that for all  $x \in \mathbb{R}^d$  it holds that

$$(\mathcal{R}(\mathcal{g}_2 \bullet \mathbb{I}_1 \bullet \mathcal{g}_1))(x) = [\mathcal{R}(\mathcal{g}_2) \circ \mathcal{R}(\mathcal{g}_1)](x) = 2^L (2^{-L}(\mathcal{R}(\mathcal{f}))(x)) = (\mathcal{R}(\mathcal{f}))(x)$$
(4.15)

(cf. Definitions 2.6 and 2.8). Furthermore, observe that item (II), item (B), Proposition 2.10, and Proposition 2.7 demonstrate that

$$\mathcal{L}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) = \mathcal{L}(\boldsymbol{g}_2) + \mathcal{L}(\mathbb{I}_1) + \mathcal{L}(\boldsymbol{g}_1) - 2 = (L+1) + 2 + L - 2 = 2L + 1.$$
(4.16)

Combining this, item (IV), item (B), Lemma 2.11, and Proposition 2.7 ensure that for all  $k \in \mathbb{N}_0 \cap [0, 2L+1]$  it holds that

$$\mathbb{D}_{k}(\mathscr{Q}_{2} \bullet \mathbb{I}_{1} \bullet \mathscr{Q}_{1}) = \begin{cases} \mathbb{D}_{k}(\mathscr{P}) & : k \in \mathbb{N}_{0} \cap [0, L) \\ 2 & : k = L \\ 4 & : k \in \mathbb{N} \cap (L, 2L+1) \\ 1 & : k = 2L+1. \end{cases}$$
(4.17)

This, (4.16), and the fact that  $\mathbb{D}_L(\not l) = 1$  imply that

$$\mathcal{P}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) = \sum_{k=1}^{2L+1} \mathbb{D}_{k}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1})(\mathbb{D}_{k-1}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) + 1)$$

$$= \left[\sum_{k=1}^{L-1} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f}) + 1)\right] + 2(\mathbb{D}_{L-1}(\boldsymbol{f}) + 1) + 4(2+1)$$

$$+ (L-1)(4(4+1)) + 1(4+1)$$

$$= \left[\sum_{k=1}^{L} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f}) + 1)\right] + \mathbb{D}_{L-1}(\boldsymbol{f}) + 1 + 12 + 20L - 20 + 5$$

$$\leq \mathcal{P}(\boldsymbol{f}) + \mathbb{D}_{L-1}(\boldsymbol{f}) + 20L \leq 2\mathcal{P}(\boldsymbol{f}) + 20L.$$
(4.18)

Moreover, note that item (III), item (C), and Proposition 2.17 shows that

$$\mathcal{S}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) = \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \le \max\{1, 2^{-1}\mathcal{S}(\boldsymbol{f})\}.$$
(4.19)

Combining this, (4.15), (4.16), (4.17), and (4.18) establishes items (i), (ii), (iii), (iv), and (v). The proof of Lemma 4.3 is thus complete.
**Corollary 4.4.** Let  $\not{\ell} \in \mathbb{N}$ ,  $d \in \mathbb{N}$  satisfy  $\mathcal{R}(\not{\ell}) \in C(\mathbb{R}^d, \mathbb{R})$  (cf. Definitions 2.1 and 2.3). Then there exists  $g \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(q) = \mathcal{R}(\mathbf{1})$ ,
- (ii) it holds that  $\mathcal{L}(q) = 4\mathcal{L}(\mathbf{f}) + 3$ ,
- (iii) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g)]$  that

$$\mathbb{D}_{k}(\boldsymbol{g}) = \begin{cases} \mathbb{D}_{k}(\boldsymbol{f}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\boldsymbol{f})) \\ 2 & : k \in \{\mathcal{L}(\boldsymbol{f}), 2\mathcal{L}(\boldsymbol{f}) + 1\} \\ 4 & : k \in \mathbb{N} \cap (\mathcal{L}(\boldsymbol{f}), 2\mathcal{L}(\boldsymbol{f}) + 1) \\ 4 & : k \in \mathbb{N} \cap (2\mathcal{L}(\boldsymbol{f}) + 1, \mathcal{L}(\boldsymbol{g})) \\ 1 & : k = \mathcal{L}(\boldsymbol{g}), \end{cases}$$
(4.20)

- (iv) it holds that  $\mathcal{P}(g) \leq \mathcal{P}(f) + \mathbb{D}_{\mathcal{H}(f)}(f) + 60\mathcal{L}(f) + 24$ , and
- (v) it holds that  $\mathcal{S}(q) \leq \max\{1, 2^{-2}\mathcal{S}(p)\}$
- (cf. Definition 2.13).

Proof of Corollary 4.4. Observe that Lemma 4.3 (applied with  $\not f \curvearrowleft \not f$ ,  $d \backsim d$  in the notation of Lemma 4.3) implies that there exist  $g \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}^d$  that it holds that  $\mathcal{R}(q) = \mathcal{R}(p)$ ,
- (II) it holds that  $\mathcal{L}(q) = 2\mathcal{L}(p) + 1$ ,
- (III) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(q)]$  that

$$\mathbb{D}_{k}(\boldsymbol{g}) = \begin{cases} \mathbb{D}_{k}(\boldsymbol{f}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\boldsymbol{f})) \\ 2 & : k = \mathcal{L}(\boldsymbol{f}) \\ 4 & : k \in \mathbb{N} \cap (\mathcal{L}(\boldsymbol{f}), \mathcal{L}(\boldsymbol{g})) \\ 1 & : k = \mathcal{L}(\boldsymbol{g}), \end{cases}$$
(4.21)

- (IV) it holds that  $\mathcal{P}(q) \leq \mathcal{P}(\boldsymbol{\ell}) + \mathbb{D}_{\mathcal{H}(\boldsymbol{\ell})}(\boldsymbol{\ell}) + 20\mathcal{L}(\boldsymbol{\ell})$ , and
- (V) it holds that  $\mathcal{S}(q) \leq \max\{1, 2^{-1}\mathcal{S}(p)\}$

(cf. Definition 2.13). Note that Lemma 4.3 (applied with  $\not l \curvearrowleft g$ ,  $d \backsim d$  in the notation of Lemma 4.3) implies that there exist  $\not h \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(\mathscr{h}) = \mathcal{R}(\mathscr{g}) = \mathcal{R}(\mathscr{f}),$
- (II) it holds that  $\mathcal{L}(\hbar) = 2(2\mathcal{L}(\not l) + 1) + 1 = 4\mathcal{L}(\not l) + 3$ ,

(III) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathscr{R})]$  that

$$\mathbb{D}_{k}(\mathscr{H}) = \begin{cases} \mathbb{D}_{k}(\mathscr{f}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\mathscr{f})) \\ 2 & : k = \mathcal{L}(\mathscr{f}) \\ 4 & : k \in \mathbb{N} \cap (\mathcal{L}(\mathscr{f}), 2\mathcal{L}(\mathscr{f}) + 1) \\ 2 & : k = 2\mathcal{L}(\mathscr{f}) + 1 \\ 4 & : k \in \mathbb{N} \cap (2\mathcal{L}(\mathscr{f}) + 1, \mathcal{L}(\mathscr{H})) \\ 1 & : k = \mathcal{L}(\mathscr{H}), \end{cases}$$
(4.22)

(IV) it holds that  $\mathcal{P}(\mathscr{h}) \leq \mathcal{P}(\mathscr{f}) + \mathbb{D}_{\mathcal{H}(\mathscr{f})}(\mathscr{f}) + 60\mathcal{L}(\mathscr{f}) + 24$ , and

(V) it holds that  $\mathcal{S}(\mathscr{h}) \leq \max\{1, 2^{-2}\mathcal{S}(\mathscr{f})\}.$ 

Observe that item (I), item (II), item (III), item (IV), and item (V) establish items (i), (ii), (iii), (iv), and (v). The proof of Corollary 4.4 is thus complete.  $\Box$ 

## 4.2 One-dimensional scaling ANNs

**Lemma 4.5.** Let  $\beta \in (0, \infty)$ ,  $B, n \in \mathbb{N}$ . Then there exists  $\not \in \mathbb{N}$  such that

- (i) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(\mathcal{I}))(x) = (B\beta)^n x$ ,
- (ii) it holds that  $\mathcal{D}(\mathbf{f}) = (1, 2B, 2B, \dots, 2B, 1) \in \mathbb{N}^{n+2}$ ,
- (iii) it holds that  $\mathbb{S}_0(\mathbf{f}) = 1$ ,  $\mathbb{S}_1(\mathbf{f}) = \beta$ , and  $\mathcal{S}(\mathbf{f}) = \max\{1, \beta\}$ , and
- (iv) it holds that  $\mathcal{P}(\mathbf{f}) = (4n-4)B^2 + (2n+4)B + 1$
- (cf. Definitions 2.1, 2.3, and 2.13).

Proof of Lemma 4.5. Throughout this proof let  $W_1 \in \mathbb{R}^{2B \times 1}$ ,  $W_2 \in \mathbb{R}^{1 \times 2B}$ ,  $W_3 \in \mathbb{R}^{2B \times 2B}$  satisfy

$$W_{1} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix}, \qquad W_{2} = \begin{pmatrix} \beta & -\beta & \beta & -\beta & \cdots & \beta & -\beta \end{pmatrix}, \quad \text{and} \quad W_{3} = W_{1}W_{2}, \quad (4.23)$$

and let  $\not{\ell}_k \in \mathbb{N}$ ,  $k \in \mathbb{N}$ , satisfy for all  $k \in \mathbb{N}$  that  $\not{\ell}_1 = ((W_1, 0), (W_2, 0)) \in \mathbb{N}$ , and  $\not{\ell}_{k+1} = \not{\ell}_k \bullet \not{\ell}_1$  (cf. Definitions 2.1 and 2.8). Note that (4.23) implies that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathcal{L}_1))(x) = B(\beta(\mathfrak{R}(x)) - \beta(\mathfrak{R}(-x))) = B\beta x.$$
(4.24)

This and Proposition 2.10 demonstrate that for all  $x \in \mathbb{R}$ ,  $k \in \mathbb{N}$  with  $\forall y \in \mathbb{R}$ :  $(\mathcal{R}(\mathcal{J}_k))(y) = (B\beta)^k y$  it holds that

$$(\mathcal{R}(\mathcal{J}_{k+1}))(x) = (\mathcal{R}(\mathcal{J}_k \bullet \mathcal{J}_1))(x) = (\mathcal{R}(\mathcal{J}_k))\big((\mathcal{R}(\mathcal{J}_1))(x)\big) = (\mathcal{R}(\mathcal{J}_k))(B\beta x) = (B\beta)^{k+1}x.$$
(4.25)

Combining this and (4.24) with induction ensures that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathcal{F}_n))(x) = (B\beta)^n x. \tag{4.26}$$

Furthermore, observe that (2.10) and (4.23) imply that

$$\boldsymbol{f}_{2} = \boldsymbol{f}_{1} \bullet \boldsymbol{f}_{1} = \left( (W_{1}, 0), (W_{1}W_{2}, W_{1} \cdot 0 + 0), (W_{2}, 0) \right) = \left( (W_{1}, 0), (W_{3}, 0), (W_{2}, 0) \right).$$
(4.27)

Combining this, (2.10), and (4.23) with induction demonstrates that

$$\mathcal{F}_{n} = \left( (W_{1}, 0), (W_{3}, 0), (W_{3}, 0), \dots, (W_{3}, 0), (W_{2}, 0) \right) \\
\in \left( (\mathbb{R}^{2B \times 1} \times \mathbb{R}^{2B}) \times \left( \times_{k=1}^{n-1} (\mathbb{R}^{2B \times 2B} \times \mathbb{R}^{2B}) \right) \times (\mathbb{R}^{1 \times 2B} \times \mathbb{R}^{1}) \right).$$
(4.28)

This and (4.23) show that

$$\mathbb{S}_0(\mathcal{F}_n) = \mathbb{S}_0(\mathcal{F}_1) = 1, \quad \mathbb{S}_1(\mathcal{F}_n) = \mathbb{S}_1(\mathcal{F}_1) = \beta, \quad \text{and} \quad \mathcal{S}(\mathcal{F}_n) = \mathcal{S}(\mathcal{F}_1) = \max\{1, \beta\} \quad (4.29)$$

Moreover, note that (4.28) ensures that

$$\mathcal{D}(\mathscr{J}_n) = (1, 2B, 2B, \dots, 2B, 1) \in \mathbb{N}^{n+2}.$$
(4.30)

Hence we obtain that

$$\mathcal{P}(\mathcal{L}_n) = \sum_{k=1}^{n+1} \mathbb{D}_k(\mathbb{D}_{k-1}+1) = 2B(1+1) + (n-1)(2B(2B+1)) + 1(2B+1)$$
  
= 4(n-1)B<sup>2</sup> + (4+2(n-1)+2)B + 1  
= (4n-4)B<sup>2</sup> + (2n+4)B + 1. (4.31)

(cf. Definitions 2.2, 2.3, and 2.13). Combining this (4.26), (4.29), (4.30), and (4.28) establishes items (i), (ii), (iii), and (iv). The proof of Lemma 4.5 is thus complete.

**Corollary 4.6.** Let  $\beta \in \mathbb{R} \setminus \{0\}$ ,  $L \in \mathbb{N}_0$  satisfy  $L \ge \log_2(|\beta|)$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

- (i) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(\mathcal{I}))(x) = \beta x$ ,
- (*ii*) it holds that  $\mathcal{D}(\mathbf{f}) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{L+2}$ ,
- (iii) it holds that  $\mathbb{S}_0(\mathbf{f}) \leq 1$ ,  $\mathbb{S}_1(\mathbf{f}) \leq 2$ , and  $\mathcal{S}(\mathbf{f}) \leq 2$ , and
- (iv) it holds that  $\mathcal{P}(\mathcal{L}) \leq 6 \max\{L, 1\} + 1$

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Corollary 4.6. Throughout this proof assume w.l.o.g. that  $L = \min(\mathbb{N}_0 \cap [\log_2(|\beta|), \infty))$ (cf. Corollary 4.1) and  $|\beta| > 1$  (otherwise consider  $((\beta), 0) \in (\mathbb{R}^{1 \times 1} \times \mathbb{R}) \subseteq \mathbb{N}$ ), and let  $g_2 \in \mathbb{N}$  satisfy

$$g_2 = \left( \left( \frac{\beta}{|\beta|} \right), 0 \right) \in \left( \mathbb{R}^{1 \times 1} \times \mathbb{R}^1 \right)$$
(4.32)

(cf. Definition 2.1). Observe that Lemma 4.5 (applied with  $\beta \curvearrowleft |\beta|^{\frac{1}{L}}$ ,  $n \curvearrowleft L$ ,  $B \curvearrowleft 1$  in the notation of Lemma 4.5) shows that there exists  $g_1 \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_1))(x) = |\beta|x$ ,
- (II) it holds that  $\mathcal{D}(g_1) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{L+2}$ ,
- (III) it holds that  $\mathbb{S}_0(\mathfrak{g}_1) = 1$ ,  $\mathbb{S}_1(\mathfrak{g}_1) = |\beta|^{\frac{1}{L}}$ , and  $\mathcal{S}(\mathfrak{g}_1) = |\beta|^{\frac{1}{L}}$ , and
- (IV) it holds that  $\mathcal{P}(g_1) = 6L + 1$

(cf. Definitions 2.3 and 2.13). Note that (4.32), item (II), and Lemma 2.11 demonstrate that

$$\mathcal{D}(\boldsymbol{g}_2 \bullet \boldsymbol{g}_1) = \mathcal{D}(\boldsymbol{g}_1) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{L+2} \quad \text{and} \quad \mathcal{P}(\boldsymbol{g}_2 \bullet \boldsymbol{g}_1) = \mathcal{P}(\boldsymbol{g}_1) = 6L + 1 \quad (4.33)$$

(cf. Definition 2.8). Furthermore, observe that (4.32), item (I), and Proposition 2.10 imply that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathcal{g}_2 \bullet \mathcal{g}_1))(x) = (\mathcal{R}(\mathcal{g}_2))(\mathcal{R}(\mathcal{g}_1)(x)) = \frac{\beta}{|\beta|}(|\beta|x) = \beta x.$$
(4.34)

Moreover, note that (2.10), (4.32), item (III), Lemma 2.16, and the fact that  $|\beta|^{\frac{1}{L}} \leq 2$  imply that for all  $x \in \mathbb{R}$  it holds that  $\mathbb{S}_0(g_2 \bullet g_1) = \mathbb{S}_0(g_1) = 1$ ,

$$\mathbb{S}_{1}(\boldsymbol{g}_{2} \bullet \boldsymbol{g}_{1}) = \left|\frac{\beta}{|\beta|}\right| \mathbb{S}_{1}(\boldsymbol{g}_{1}) \leq 2, \quad \text{and} \quad \mathcal{S}(\boldsymbol{g}_{2} \bullet \boldsymbol{g}_{1}) \leq \max\{\mathcal{S}(\boldsymbol{g}_{1}), \mathbb{S}_{1}(\boldsymbol{g}_{2} \bullet \boldsymbol{g}_{1})\} \leq 2.$$

$$(4.35)$$

Combining this and (4.33) with (4.34) establishes items (i), (ii), (iii), and (iv). The proof of Corollary 4.6 is thus complete.

## 4.3 Upper bounds for approximations of product functions

**Lemma 4.7.** Let  $N \in \mathbb{N}$ . Then there exists  $\not \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\not) \in C(\mathbb{R}, \mathbb{R})$ ,
- (*ii*) it holds that  $\sup_{x \in [0,1]} |x^2 (\mathcal{R}(\mathbf{f}))(x)| \le 4^{-N-1}$ ,
- (iii) it holds for all  $x \in \mathbb{R} \setminus [0, 1]$  that  $(\mathcal{R}(\mathbf{f}))(x) = \mathfrak{R}(x)$ ,
- (iv) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(\mathcal{L}))(x) (\mathcal{R}(\mathcal{L}))(y)| \le 2|x-y|$ ,
- (v) it holds that  $\mathcal{D}(\mathbf{f}) = (1, 4, 4, \dots, 4, 1) \in \mathbb{N}^{N+2}$ , and
- (vi) it holds that  $\mathcal{S}(\mathbf{f}) \leq 4$

(cf. Definitions 2.1, 2.2, 2.3, and 2.13).

Proof of Lemma 4.7. Observe that Lemma 5.1 and Lemma 5.2 in Grohs et al. [13] proves that there exists  $\ell \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(\not \in) \in C(\mathbb{R}, \mathbb{R})$ ,
- (II) it holds that  $\sup_{x \in [0,1]} \left| x^2 (\mathcal{R}(\mathbf{\ell}))(x) \right| \le 4^{-N-1}$ ,
- (III) it holds for all  $x \in \mathbb{R} \setminus [0, 1]$  that  $(\mathcal{R}(\not l))(x) = \mathfrak{R}(x)$ ,

(IV) it holds for all  $k \in \{0, 1, \dots, 2^N - 1\}, x \in \left[\frac{k}{2^N}, \frac{k+1}{2^N}\right)$  that  $(\mathcal{R}(\mathbf{f}))(x) = \left(\frac{2N+1}{2^N}\right)x - \frac{k^2+k}{2^{2N}}$ ,

- (V) it holds that  $\mathcal{D}(\mathbf{\ell}) = (1, 4, 4, \dots, 4, 1) \in \mathbb{N}^{N+2}$ , and
- (VI) it holds that  $\mathcal{S}(\not l) \leq 4$

(cf. Definitions 2.1, 2.2, 2.3, and 2.13). Note that item (I), item (III), item (IV), and the triangle inequality ensure that for all  $x, y \in \mathbb{R}$  it holds that

$$\left| (\mathcal{R}(\boldsymbol{\ell}))(x) - (\mathcal{R}(\boldsymbol{\ell}))(y) \right| \le \max\left\{ 1, \frac{2^{N+1}-1}{2^N} \right\} |x-y| \le 2|x-y|.$$
(4.36)

Combining this with items (I), (II), (III), (V), and (VI) establishes items (i), (ii), (iii), (iv), (v), and (vi). The proof of Lemma 4.7 is thus complete.  $\Box$ 

**Definition 4.8** (Ceiling of real numbers). We denote by  $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$  the function which satisfies for all  $x \in \mathbb{R}$  that  $\lceil x \rceil = \min(\mathbb{Z} \cap [x, \infty))$ .

**Lemma 4.9.** Let  $N \in \mathbb{N}$ ,  $R \in (1, \infty)$ . Then there exists  $f \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\not e) \in C(\mathbb{R}, \mathbb{R})$ ,
- (*ii*) it holds that  $\sup_{x \in [-R,R]} |x^2 (\mathcal{R}(\mathbf{f}))(x)| \le R^2 4^{-N-1}$ ,
- (iii) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(\mathbf{f}))(x) (\mathcal{R}(\mathbf{f}))(y)| \le 2R|x-y|$ ,
- (iv) it holds that  $\mathcal{L}(\mathcal{L}) = N + \lceil \log_2(R) \rceil + 4$ ,
- (v) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{J})]$  that

$$\mathbb{D}_{k}(\mathscr{I}) = \begin{cases} 1 & : k = 0 \\ 2 & : k \in \mathbb{N} \cap (0, 2] \\ 4 & : k \in \mathbb{N} \cap (2, N + 2] \\ 2 & : k \in \mathbb{N} \cap (N + 2, N + \lceil \log_{2}(R) \rceil + 4) \\ 1 & : k = N + \lceil \log_{2}(R) \rceil + 4, \end{cases}$$
(4.37)

and

- (vi) it holds that  $\mathcal{S}(\not e) \leq 4$
- (cf. Definitions 2.1, 2.3, 2.13, and 4.8).

*Proof.* Throughout this proof let  $n \in \mathbb{N}$  satisfy  $n = \lfloor \log_2(R) \rfloor$  and let  $g_1 \in \mathbb{N}$ , satisfy

$$g_1 = \left( \left( \begin{pmatrix} R^{-1} \\ -R^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \left( \begin{pmatrix} 1 & 1 \end{pmatrix}, 0 \right) \right) \in \left( \left( \mathbb{R}^{2 \times 1} \times \mathbb{R}^2 \right) \times \left( \mathbb{R}^{1 \times 2} \times \mathbb{R} \right) \right)$$
(4.38)

(cf. Definitions 2.1 and 4.8). Furthermore, observe that Lemma 4.7 (applied with  $N \curvearrowleft N$  in the notation of Lemma 4.7) shows that there exists  $g_2 \in \mathbf{N}$  which satisfies that

(I) it holds that  $\mathcal{R}(\mathfrak{g}_2) \in C(\mathbb{R}, \mathbb{R})$ ,

- (II) it holds that  $\sup_{x \in [0,1]} \left| x^2 (\mathcal{R}(\mathcal{G}_2))(x) \right| \le 4^{-N-1}$ ,
- (III) it holds for all  $x \in \mathbb{R} \setminus [0, 1]$  that  $(\mathcal{R}(g_2))(x) = \mathfrak{R}(x)$ ,

(IV) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(g_2))(x) - (\mathcal{R}(g_2))(y)| \le 2|x-y|,$ 

- (V) it holds that  $\mathcal{D}(g_2) = (1, 4, 4, \dots, 4, 1) \in \mathbb{N}^{N+2}$ , and
- (VI) it holds that  $\mathcal{S}(q_2) \leq 4$

(cf. Definitions 2.2, 2.3, and 2.13). Note that Lemma 4.5 (applied with  $\beta \curvearrowleft R^{\frac{2}{n}}, B \curvearrowleft 1, n \curvearrowleft n$  in the notation of Lemma 4.5) shows that there exists  $g_3 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_3))(x) = R^2 x$ ,
- (B) it holds that  $\mathcal{D}(g_3) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{n+2}$ , and
- (C) it holds that  $\mathbb{S}_0(\mathfrak{g}_3) = 1$ ,  $\mathbb{S}_1(\mathfrak{g}_3) = R^{\frac{2}{n}}$ , and  $\mathcal{S}(\mathfrak{g}_3) = R^{\frac{2}{n}}$ .

Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{\ell} = \boldsymbol{g}_3 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1 \tag{4.39}$$

(cf. Definitions 2.6 and 2.8). Observe that (4.38), item (V), item (B), Proposition 2.10, Lemma 2.11, and Proposition 2.7 demonstrate that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{F})]$  it holds that

$$\mathcal{L}(\mathscr{L}) = N + n + 4 \quad \text{and} \quad \mathbb{D}_{k}(\mathscr{L}) = \begin{cases} 1 & : k = 0 \\ 2 & : k \in \mathbb{N} \cap (0, 2] \\ 4 & : k \in \mathbb{N} \cap (2, N + 2] \\ 2 & : k \in \mathbb{N} \cap (N + 2, N + n + 4) \\ 1 & : k = N + n + 4. \end{cases}$$
(4.40)

Moreover, note that (4.38), (4.39), item (A), Proposition 2.10, and Proposition 2.7 prove that for all  $x \in \mathbb{R}$  it holds that

$$\mathcal{R}(\mathbf{f}) \in C(\mathbb{R}, \mathbb{R})$$
 and  $(\mathcal{R}(\mathbf{f}))(x) = R^2 \left[ (\mathcal{R}(\mathbf{g}_2)) \left( \frac{|x|}{R} \right) \right].$  (4.41)

Combining this with item (II) demonstrates that for all  $x \in [-R, R]$  it holds that

$$|x^{2} - (\mathcal{R}(\boldsymbol{\ell}))(x)| = \left| R^{2} \left[ \frac{|x|}{R} \right]^{2} - R^{2} \left[ (\mathcal{R}(\boldsymbol{g}_{2})) \left( \frac{|x|}{R} \right) \right] \right| \le R^{2} 4^{-N-1}.$$
(4.42)

In addition, observe that (4.41) and item (IV) imply that for all  $x, y \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\boldsymbol{f}))(x) - (\mathcal{R}(\boldsymbol{f}))(y)| = \left| R^2 \left[ (\mathcal{R}(\boldsymbol{g}_2)) \left( \frac{|x|}{R} \right) \right] - R^2 \left[ (\mathcal{R}(\boldsymbol{g}_2)) \left( \frac{|y|}{R} \right) \right] \right|$$
  
$$\leq 2R^2 \left| \frac{|x|}{R} - \frac{|y|}{R} \right| \leq 2R|x - y|.$$
(4.43)

Furthermore, note that (4.38), (4.39), item (VI), item (C), Proposition 2.18, and the fact that  $1 < R \leq 2^n$  ensure that

$$\mathcal{S}(\boldsymbol{f}) \leq \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \leq \max\{R^{\frac{2}{n}}, 4, 1\} = 4.$$

$$(4.44)$$

Combining this with (4.40), (4.41), (4.42), and (4.43), establishes items (i), (ii), (iii), (iv), (v), and (vi). The proof of Lemma 4.9 is thus complete.

**Lemma 4.10.** Let  $N \in \mathbb{N}$ ,  $R \in (1, \infty)$ . Then there exists  $\not \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\not l) \in C(\mathbb{R}^2, \mathbb{R})$ ,
- (*ii*) it holds that  $\sup_{x,y\in[-R,R]} |xy (\mathcal{R}(\mathcal{L}))(x,y)| \le 3R^2 2^{-2N-1}$ ,
- (iii) it holds for all  $x, y \in \mathbb{R}^2$  that  $|(\mathcal{R}(\mathbf{f}))(x) (\mathcal{R}(\mathbf{f}))(y)| \leq \sqrt{32}R||x-y||_2$ ,
- (iv) it holds that  $\mathcal{L}(\mathbf{f}) = N + \lceil \log_2(R) \rceil + 7$ ,
- (v) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{L})]$  that

$$\mathbb{D}_{k}(\mathscr{L}) = \begin{cases} 2 & : k = 0 \\ 6 & : k \in \mathbb{N} \cap (0, 3] \\ 12 & : k \in \mathbb{N} \cap (3, N+3] \\ 6 & : k \in \mathbb{N} \cap (N+3, N+\lceil \log_{2}(R) \rceil + 7) \\ 1 & : k = N + \lceil \log_{2}(R) \rceil + 7, \end{cases}$$
(4.45)

and

- (vi) it holds that  $\mathcal{S}(\not e) \leq 4$
- (cf. Definitions 2.1, 2.3, 2.13, 3.14, and 4.8).

Proof of Lemma 4.10. Throughout this proof let  $q_1 \in (\mathbb{R}^{3 \times 2} \times \mathbb{R}^3) \subseteq \mathbb{N}, q_3 \in (\mathbb{R}^{1 \times 3} \times \mathbb{R}) \subseteq \mathbb{N}$  satisfy

$$g_1 = \left( \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \quad \text{and} \quad g_3 = \left( \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, 0 \right) \quad (4.46)$$

(cf. Definition 2.1). Observe that (4.46) ensures that

$$\mathcal{D}(\boldsymbol{g}_1) = (2,3), \ \mathcal{D}(\boldsymbol{g}_3) = (3,1), \ \mathcal{R}(\boldsymbol{g}_1) \in C(\mathbb{R}^2, \mathbb{R}^3), \ \text{and} \ \mathcal{R}(\boldsymbol{g}_3) \in C(\mathbb{R}^3, \mathbb{R})$$
(4.47)

(cf. Definition 2.3). Furthermore, note that (4.46) implies that for all  $x, y, z \in \mathbb{R}$  it holds that

$$(\mathcal{R}(g_1))(x,y) = (x+y,x,y)$$
 and  $(\mathcal{R}(g_3))(x,y,z) = \frac{x-y-z}{2}$ . (4.48)

Observe that Lemma 4.9 (applied with  $N \curvearrowleft N$ ,  $R \curvearrowleft 2R$  in the notation of Lemma 4.9) shows that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(\mathfrak{g}_2) \in C(\mathbb{R}, \mathbb{R})$ ,
- (II) it holds that  $\sup_{x \in [-2R,2R]} \left| x^2 (\mathcal{R}(g_2))(x) \right| \le R^2 4^{-N}$ ,
- (III) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(g_2))(x) (\mathcal{R}(g_2))(y)| \le 4R|x-y|,$
- (IV) it holds that  $\mathcal{L}(g_2) = N + \lceil \log_2(2R) \rceil + 4$ ,

(V) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g_2)]$  that

$$\mathbb{D}_{k}(\boldsymbol{g}_{2}) = \begin{cases} 1 & : k = 0 \\ 2 & : k \in \mathbb{N} \cap (0, 2] \\ 4 & : k \in \mathbb{N} \cap (2, N + 2] \\ 2 & : k \in \mathbb{N} \cap (N + 2, N + \lceil \log_{2}(2R) \rceil + 4) \\ 1 & : k = N + \lceil \log_{2}(2R) \rceil + 4, \end{cases}$$
(4.49)

and

(VI) it holds that  $\mathcal{S}(g_2) \leq 4$ 

(cf. Definitions 2.13 and 4.8). Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{\ell} = \boldsymbol{g}_3 \bullet \mathbb{I}_3 \bullet (\mathbf{P}_3(\boldsymbol{g}_2, \boldsymbol{g}_2, \boldsymbol{g}_2)) \bullet \mathbb{I}_3 \bullet \boldsymbol{g}_1 \tag{4.50}$$

(cf. Definitions 2.4, 2.6, and 2.8). Note that (4.47), (4.50), item (IV), Proposition 2.10, Proposition 2.5, and Proposition 2.7 ensure that

$$\mathcal{L}(\boldsymbol{\ell}) = \mathcal{L}(\boldsymbol{g}_3) + \mathcal{L}(\mathbb{I}_3) + \mathcal{L}(\mathbf{P}_3(\boldsymbol{g}_2, \boldsymbol{g}_2, \boldsymbol{g}_2)) + \mathcal{L}(\mathbb{I}_3) + \mathcal{L}(\boldsymbol{g}_1) - 4$$
  
= 1 + 2 +  $\mathcal{L}(\boldsymbol{g}_2) + 2 + 1 - 4$   
= N +  $\lceil \log_2(2R) \rceil + 6 = N + \lceil \log_2(R) \rceil + 7.$  (4.51)

This, (4.47), (4.50), item (V), Lemma 2.11, Proposition 2.5, and Proposition 2.7 ensure that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{L})]$  it holds that

$$\mathbb{D}_{k}(\mathscr{I}) = \begin{cases} 2 & : k = 0 \\ 6 & : k \in \mathbb{N} \cap (0, 3] \\ 12 & : k \in \mathbb{N} \cap (3, N+3] \\ 6 & : k \in \mathbb{N} \cap (N+3, N+\lceil \log_{2}(R) \rceil + 7) \\ 1 & : k = N + \lceil \log_{2}(R) \rceil + 7, \end{cases}$$
(4.52)

Next observe that (4.48), (4.50), Proposition 2.10, and Proposition 2.7 prove that for all  $x, y \in \mathbb{R}$  it holds that  $\mathcal{R}(\mathscr{L}) \in C(\mathbb{R}^2, \mathbb{R})$  and

$$(\mathcal{R}(\boldsymbol{\ell}))(x,y) = \frac{1}{2} \big[ (\mathcal{R}(\boldsymbol{g}_2))(x+y) - (\mathcal{R}(\boldsymbol{g}_2))(x) - (\mathcal{R}(\boldsymbol{g}_2))(y) \big].$$
(4.53)

This and item (II) demonstrate that for all  $x, y \in [-R, R]$  it holds that

$$\begin{aligned} |xy - (\mathcal{R}(\mathscr{I}))(x,y)| \\ &= \frac{1}{2} |(x+y)^2 - x^2 - y^2 - (\mathcal{R}(\mathscr{Q}_2))(x+y) + (\mathcal{R}(\mathscr{Q}_2))(x) + (\mathcal{R}(\mathscr{Q}_2))(y)| \\ &\leq \frac{1}{2} |(x+y)^2 - (\mathcal{R}(\mathscr{Q}_2))(x+y)| + \frac{1}{2} |x^2 - (\mathcal{R}(\mathscr{Q}_2))(x)| + \frac{1}{2} |y^2 - (\mathcal{R}(\mathscr{Q}_2))(y)| \\ &\leq \frac{3}{2} (R^2 4^{-N}) = 3R^2 2^{-2N-1}. \end{aligned}$$

$$(4.54)$$

Moreover, note that (4.53) and item (III) show that for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  it holds that

$$\begin{aligned} |(\mathcal{R}(\boldsymbol{f}))(x_{1}, x_{2}) - (\mathcal{R}(\boldsymbol{f}))(y_{1}, y_{2})| \\ &\leq \frac{1}{2} \big( |(\mathcal{R}(\boldsymbol{g}_{2}))(x_{1} + x_{2}) - (\mathcal{R}(\boldsymbol{g}_{2}))(y_{1} + y_{2})| \\ &+ |(\mathcal{R}(\boldsymbol{g}_{2}))(x_{1}) - (\mathcal{R}(\boldsymbol{g}_{2}))(y_{1})| + |(\mathcal{R}(\boldsymbol{g}_{2}))(x_{2}) - (\mathcal{R}(\boldsymbol{g}_{2}))(y_{2})| \big) \\ &\leq 2R \big( |(x_{1} + x_{2}) - (y_{1} + y_{2})| + |x_{1} - y_{1}| + |x_{2} - y_{2}| \big) \\ &\leq 4R(|x_{1} - y_{1}| + |x_{2} - y_{2}|) \leq \sqrt{32}R ||(x_{1} - y_{1}, x_{2} - y_{2})||_{2} \end{aligned}$$
(4.55)

(cf. Definition 3.14). In addition, observe that (4.46), (4.50), item (VI), Lemma 2.14, and Proposition 2.18 imply that

$$\mathcal{S}(\boldsymbol{\ell}) = \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\mathbf{P}_3(\boldsymbol{g}_2, \boldsymbol{g}_2, \boldsymbol{g}_2)), \mathcal{S}(\boldsymbol{g}_1)\} = \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \leq \max\{\frac{1}{2}, 4, 1\} = 4.$$

$$(4.56)$$

This, (4.52), (4.53), (4.54), and (4.55) establish items (i), (ii), (iii), (iv), (v), and (vi). The proof of Lemma 4.10 is thus complete.

**Lemma 4.11.** Let  $L \in \mathbb{R}$ ,  $d \in \mathbb{N}$ ,  $m_1, m_2, \ldots, m_d \in \mathbb{N}$ , let  $g_i \in C(\mathbb{R}^{m_i}, \mathbb{R})$ ,  $i \in \{1, 2, \ldots, d\}$ , satisfy for all  $i \in \{1, 2, \ldots, d\}$ ,  $x, y \in \mathbb{R}^{m_i}$  that  $|g_i(x) - g_i(y)| \leq L ||x - y||_2$ , and let  $f \in C(\mathbb{R}^{[\sum_{i=1}^d m_i]}, \mathbb{R}^d)$  satisfy for all  $x = (x_1, \ldots, x_d) \in (\times_{i=1}^d \mathbb{R}^{m_i})$  that  $f(x) = (g_1(x_1), g_2(x_2), \ldots, g_d(x_d))$ . Then it holds for all  $x, y \in \mathbb{R}^{[\sum_{i=1}^d m_i]}$  that

$$\|f(x) - f(y)\|_2 \le L \|x - y\|_2 \tag{4.57}$$

(cf. Definition 3.14).

*Proof of Lemma 4.11.* Note that Beneventano et al. [3, Lemma 3.22] establishes (4.57). The proof of Lemma 4.11 is thus complete.  $\Box$ 

**Lemma 4.12.** Let  $d, N \in \mathbb{N}$ ,  $R \in (1, \infty)$ . Then there exists  $f \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\mathscr{L}) \in C(\mathbb{R}^{2d}, \mathbb{R}^d)$ ,
- (ii) it holds for all  $x = (x_1, \ldots, x_{2d}) \in [-R, R]^{2d}$  that

$$\|(x_1x_2, x_3x_4, \dots, x_{2d-1}x_{2d}) - (\mathcal{R}(\mathbf{\ell}))(x)\|_2 \le 3R^2 d^{\frac{1}{2}} 2^{-2N-1}, \tag{4.58}$$

(iii) it holds for all  $x, y \in \mathbb{R}^{2d}$  that  $\left\| (\mathcal{R}(\mathbf{f}))(x) - (\mathcal{R}(\mathbf{f}))(y) \right\|_2 \le \sqrt{32}R \|x - y\|_2$ ,

- (iv) it holds that  $\mathcal{L}(\mathcal{L}) = N + \lceil \log_2(R) \rceil + 7$ ,
- (v) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{J})]$  that

$$\mathbb{D}_{k}(\mathscr{I}) = \begin{cases} 2d & : k = 0\\ 6d & : k \in \mathbb{N} \cap (0, 3]\\ 12d & : k \in \mathbb{N} \cap (3, N+3]\\ 6d & : k \in \mathbb{N} \cap (N+3, N+\lceil \log_{2}(R) \rceil + 7)\\ d & : k = N+\lceil \log_{2}(R) \rceil + 7, \end{cases}$$
(4.59)

- (vi) it holds that  $\mathcal{P}(\mathbf{f}) = 234d^2 + 49d + N(144d^2 + 12d) + \lceil \log_2(R) \rceil (36d^2 + 6d)$ , and
- (vii) it holds that  $\mathcal{S}(\mathbf{f}) \leq 4$
- (cf. Definitions 2.1, 2.3, 2.13, 3.14, and 4.8).

Proof of Lemma 4.12. Observe that Lemma 4.10 (applied with  $N \curvearrowleft N, R \curvearrowleft R$  in the notation of Lemma 4.10) proves that there exists  $g \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(q) \in C(\mathbb{R}^2, \mathbb{R})$ ,
- (II) it holds for all  $x, y \in \mathbb{R}^2$  that  $|(\mathcal{R}(g))(x) (\mathcal{R}(g))(y)| \le \sqrt{32R} ||x y||_2$ ,
- (III) it holds that  $\sup_{x,y\in[-R,R]} |xy (\mathcal{R}(g))(x,y)| \le 3R^2 2^{-2N-1}$ ,
- (IV) it holds that  $\mathcal{L}(q) = N + \lceil \log_2(R) \rceil + 7$ ,
- (V) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g)]$  that

$$\mathbb{D}_{k}(\mathscr{G}) = \begin{cases} 2 & : k = 0 \\ 6 & : k \in \mathbb{N} \cap (0, 3] \\ 12 & : k \in \mathbb{N} \cap (3, N+3] \\ 6 & : k \in \mathbb{N} \cap (N+3, N+\lceil \log_{2}(R) \rceil + 7) \\ 1 & : k = N + \lceil \log_{2}(R) \rceil + 7, \end{cases}$$
(4.60)

and

(VI) it holds that  $\mathcal{S}(q) \leq 4$ 

(cf. Definitions 2.1, 2.3, 2.13, 3.14, and 4.8). Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{\ell} = \mathbf{P}_d(\boldsymbol{g}, \boldsymbol{g}, \dots, \boldsymbol{g}) \tag{4.61}$$

(cf. Definition 2.4). Note that (4.61), item (IV), item (V), and Proposition 2.5 ensure that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{J})]$  it holds that

$$\mathcal{L}(\boldsymbol{\ell}) = N + \lceil \log_2(R) \rceil + 7 \text{ and } \mathbb{D}_k(\boldsymbol{\ell}) = \begin{cases} 2d & : k = 0 \\ 6d & : k \in \mathbb{N} \cap (0,3] \\ 12d & : k \in \mathbb{N} \cap (3, N+3] \\ 6d & : k \in \mathbb{N} \cap (N+3, N+\lceil \log_2(R) \rceil + 7) \\ d & : k = N + \lceil \log_2(R) \rceil + 7. \end{cases}$$
(4.62)

Hence we obtain that

$$\begin{aligned} \mathcal{P}(\boldsymbol{f}) \\ &= \sum_{k=1}^{\mathcal{L}(\boldsymbol{f})} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f})+1) \\ &= \mathbb{D}_{1}(\boldsymbol{f})(\mathbb{D}_{0}(\boldsymbol{f})+1) + \mathbb{D}_{2}(\boldsymbol{f})(\mathbb{D}_{1}(\boldsymbol{f})+1) + \mathbb{D}_{3}(\boldsymbol{f})(\mathbb{D}_{2}(\boldsymbol{f})+1) + \mathbb{D}_{4}(\boldsymbol{f})(\mathbb{D}_{3}(\boldsymbol{f})+1) \\ &+ \left[\sum_{k=5}^{N+3} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f})+1)\right] + \mathbb{D}_{N+4}(\boldsymbol{f})(\mathbb{D}_{N+3}(\boldsymbol{f})+1) \\ &+ \left[\sum_{k=N+5}^{\mathcal{L}(\boldsymbol{f})-1} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f})+1)\right] + \mathbb{D}_{\mathcal{L}(\boldsymbol{f})}(\boldsymbol{f})(\mathbb{D}_{\mathcal{L}(\boldsymbol{f})-1}(\boldsymbol{f})+1) \\ &= 6d(2d+1) + 2(6d(6d+1)) + 12d(6d+1) + \left[\sum_{k=3}^{N+1} 12d(12d+1)\right] + 6d(12d+1) \\ &+ \left[\sum_{k=N+3}^{\mathcal{L}(\boldsymbol{f})-1} 6d(6d+1)\right] + d(6d+1) \\ &= (12+72+72+72+6)d^{2} + (6+12+12+6+1)d + (N-1)(144d^{2}+12d) \\ &+ (\mathcal{L}(\boldsymbol{f})-N-3)(36d^{2}+6d) \\ &= 234d^{2} + 37d + (N-1)(144d^{2}+12d) + (\lceil \log_{2}(R) \rceil + 4)(36d^{2}+6d) \\ &= 234d^{2} + 49d + N(144d^{2}+12d) + \lceil \log_{2}(R) \rceil (36d^{2}+6d). \end{aligned}$$

Furthermore, observe that (4.61), item (III), and Proposition 2.5 show that for all  $x = (x_1, \ldots, x_{2d}) \in [-R, R]^{2d}$  it holds that  $\mathcal{R}(\mathscr{I}) \in C(\mathbb{R}^{2d}, \mathbb{R}^d)$  and

$$\|(x_1x_2, x_3x_4, \dots, x_{2d-1}x_{2d}) - (\mathcal{R}(\mathscr{f}))(x)\|_2 = \left[\sum_{i=1}^d |x_{2i-1}x_{2i} - (\mathcal{R}(\mathscr{g}))(x_{2i-1}, x_{2i})|^2\right]^{\frac{1}{2}} \le \left[\sum_{i=1}^d 9R^4 2^{-4N-2}\right]^{\frac{1}{2}} = 3R^2 d^{\frac{1}{2}} 2^{-2N-1}.$$
(4.64)

Moreover, note that item (VI) and Lemma 2.14 ensure that

$$\mathcal{S}(\boldsymbol{\ell}) = \mathcal{S}(\boldsymbol{g}) \le 4. \tag{4.65}$$

Next we combine (4.61), item (II), and Proposition 2.5 with Lemma 4.11 (applied with  $L \curvearrowright \sqrt{32R}$ ,  $d \curvearrowright d$ ,  $(g_1, g_2, \ldots, g_d) \curvearrowleft (\mathcal{R}(g), \mathcal{R}(g), \ldots, \mathcal{R}(g))$ ,  $f \backsim \mathcal{R}(f)$  in the notation of Lemma 4.11) to obtain that for all  $x, y \in \mathbb{R}^{2d}$  it holds that

$$\|(\mathcal{R}(\boldsymbol{f}))(x) - (\mathcal{R}(\boldsymbol{f}))(y)\|_2 \le \sqrt{32R} \|x - y\|_2.$$
(4.66)

This, (4.62), (4.63), (4.64), and (4.65) establish items (i), (ii), (iii), (iv), (v), (vi), and (vii). The proof of Lemma 4.12 is thus complete.

**Lemma 4.13.** Let  $n \in \mathbb{N}$ ,  $d_0, d_1, \ldots, d_n \in \mathbb{N}$ ,  $L_1, L_2, \ldots, L_n, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in [0, \infty)$ , let  $D_i \subseteq \mathbb{R}^{d_{i-1}}$ ,  $i \in \{1, 2, \ldots, n\}$ , be sets, for every  $i \in \{1, 2, \ldots, n\}$  let  $f_i: D_i \to \mathbb{R}^{d_i}$  and  $g_i: \mathbb{R}^{d_{i-1}} \to \mathbb{R}^{d_i}$  satisfy for all  $x \in D_i$  that

$$\|f_i(x) - g_i(x)\|_2 \le \varepsilon_i,\tag{4.67}$$

and assume for all  $j \in \mathbb{N} \cap (0, n)$ ,  $x, y \in \mathbb{R}^{d_j}$  that

$$f_j(D_j) \subseteq D_{j+1}$$
 and  $||g_{j+1}(x) - g_{j+1}(y)||_2 \le L_{j+1} ||x - y||_2$  (4.68)

(cf. Definition 3.14). Then it holds for all  $x \in D_1$  that

$$\|(f_n \circ f_{n-1} \circ \ldots \circ f_1)(x) - (g_n \circ g_{n-1} \circ \ldots \circ g_1)(x)\|_2 \le \sum_{i=1}^n \left[ \left(\prod_{j=i+1}^n L_j\right) \varepsilon_i \right].$$
(4.69)

*Proof of Lemma 4.13.* Observe that Beneventano et al. [3, Lemma 6.5] establishes (4.69). The proof of Lemma 4.13 is thus complete.  $\Box$ 

**Lemma 4.14.** Let  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $R \in (1,\infty)$ . Then there exists  $\not \in \mathbb{N}$  such that

(i) it holds that  $\mathcal{R}(\mathcal{L}) \in C(\mathbb{R}^{(2^d)}, \mathbb{R}),$ 

(*ii*) it holds for all 
$$x = (x_1, ..., x_{2^d}) \in [-R, R]^{(2^d)}$$
 that  $\left| \prod_{i=1}^{2^d} x_i - (\mathcal{R}(\mathcal{I}))(x) \right| \leq \varepsilon$ ,

(*iii*) it holds for all 
$$x, y \in \mathbb{R}^{(2^d)}$$
 that  $|(\mathcal{R}(\mathcal{I}))(x) - (\mathcal{R}(\mathcal{I}))(y)| \le 2^{\frac{5d}{2}} R^{(2^d-1)} ||x-y||_2$ ,

- (iv) it holds that  $\mathcal{L}(\mathbf{f}) \leq d2^{d+2} + d2^d \lceil \log_2(R) \rceil \frac{d \log_2(\varepsilon)}{2}$ ,
- (v) it holds that that  $\mathbb{D}_1(\mathcal{F}) = 2^d 3$  and  $\mathbb{D}_{\mathcal{H}(\mathcal{F})}(\mathcal{F}) = 6$ ,
- (vi) it holds that  $\mathcal{P}(\mathbf{p}) \leq 2^{3d+10} + 2^{3d+8} \lceil \log_2(R) \rceil 2^{2d+7} \log_2(\varepsilon)$ , and
- (vii) it holds that  $\mathcal{S}(\not l) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, 3.14, and 4.8).

Proof of Lemma 4.14. Throughout this proof assume w.l.o.g. that d > 1, for every  $i \in \{1, 2, \ldots, d\}$ let  $N_i \in \mathbb{N}$  satisfy  $N_i = \left\lceil \frac{8d-5i}{4} + (2^{d-1}-2^{i-1}+1)\log_2(R) - \frac{1}{2}\log_2(\varepsilon) + \frac{1}{2} \right\rceil$ , for every  $i \in \{1, 2, \ldots, d\}$ let  $D_i \subseteq \mathbb{R}^{(2^{d-i+1})}$  satisfy  $D_i = \left[ -R^{(2^{i-1})}, R^{(2^{i-1})} \right]^{(2^{d-i+1})}$ , and for every  $i \in \{1, 2, \ldots, d\}$  let  $p_i \colon D_i \to \mathbb{R}^{(2^{d-i})}$  satisfy for all  $x = (x_1, x_2, \ldots, x_{2^{d-i+1}}) \in D_i$  that

$$p_i(x) = (x_1 x_2, x_3 x_4, \dots, x_{2^{d-i+1}-1} x_{2^{d-i+1}}).$$
(4.70)

Note that the fact that for all  $i \in \{1, 2, ..., d\}$  it holds that  $N_i \geq \frac{8d-5i}{4} + (2^{d-1} - 2^{i-1} + 1)\log_2(R) - \frac{1}{2}\log_2(\varepsilon) + \frac{1}{2}$  and the fact that for all  $k \in \mathbb{N}$  it holds that  $2^{-k} \leq k^{-1}$  imply that for all  $i \in \{1, 2, ..., d\}$  it holds that

$$2^{\frac{5d-5i}{2}}R^{(2^d-2^i)}3R^2d^{\frac{1}{2}}2^{-2N_i-1} \le 2^{\frac{5d-5i}{2}}R^{(2^d-2^i)}3R^2d^{\frac{1}{2}}2^{-(\frac{8d-5i}{2}+(2^d-2^i+2)\log_2(R)-\log_2(\varepsilon)+1)-1}$$

$$= 2^{-\frac{3d}{2}-2}R^{(2^d-2^i)}3R^2d^{\frac{1}{2}}R^{-(2^d-2^i+2)}\varepsilon$$

$$= 2^{-\frac{3d}{2}-2}3d^{\frac{1}{2}}\varepsilon$$

$$\le 2^{-\frac{3d}{2}}d^{\frac{1}{2}}\varepsilon \le d^{-\frac{3}{2}}d^{\frac{1}{2}}\varepsilon = d^{-1}\varepsilon.$$
(4.71)

Observe that Lemma 4.12 (applied with  $d \curvearrowleft 2^{d-i}$ ,  $N \curvearrowleft N_i$ ,  $R \curvearrowleft R^{(2^{i-1})}$ ,  $\varepsilon \backsim 2^{\frac{5i-5d}{2}}R^{(2^i-2^d)}d^{-1}\varepsilon$  for  $i \in \{1, 2, \ldots, d\}$  in the notation of Lemma 4.12) shows that for every  $i \in \{1, 2, \ldots, d\}$  there exists  $\mathscr{H}_i \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(\mathbb{A}_i) \in C(\mathbb{R}^{(2^{d-i+1})}, \mathbb{R}^{(2^{d-i})}),$
- (II) it holds that  $\sup_{x \in D_i} \|p_i(x) (\mathcal{R}(\mathscr{R}_i))(x)\|_2 \le 3R^{(2^i)}2^{\frac{d-i}{2}}2^{-2N_i-1}$ ,
- (III) it holds for all  $x, y \in \mathbb{R}^{(2^{d-i+1})}$  that  $\left\| (\mathcal{R}(\mathscr{K}_i))(x) (\mathcal{R}(\mathscr{K}_i))(y) \right\|_2 \le \sqrt{32} R^{(2^{i-1})} \|x y\|_2$ ,
- (IV) it holds that  $\mathcal{L}(\mathscr{R}_i) = N_i + \lceil 2^{i-1} \log_2(R) \rceil + 7$ ,
- (V) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathscr{K}_i)]$  that

$$\mathbb{D}_{k}(\mathscr{R}_{i}) = \begin{cases} 2^{d-i+1} & : k = 0\\ 2^{d-i+1} 3 & : k \in \mathbb{N} \cap (0,3]\\ 2^{d-i+2} 3 & : k \in \mathbb{N} \cap (3, N_{i}+3]\\ 2^{d-i+1} 3 & : k \in \mathbb{N} \cap (N_{i}+3, N_{i}+\lceil 2^{i-1}\log_{2}(R)\rceil + 7]\\ 2^{d-i} & : k = N_{i}+\lceil 2^{i-1}\log_{2}(R)\rceil + 7, \end{cases}$$

$$(4.72)$$

- (VI) it holds that  $\mathcal{P}(\aleph_i) = 2^{2d-2i}234 + 2^{d-i}49 + N_i(2^{2d-2i}144 + 2^{d-i}12) + \lceil \log_2(R) \rceil (2^{2d-2i}36 + 2^{d-i}6)$ , and
- (VII) it holds that  $\mathcal{S}(\mathscr{K}_i) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, 3.14, and 4.8). Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{\ell} = \boldsymbol{\hbar}_d \bullet \mathbb{I}_2 \bullet \boldsymbol{\hbar}_{d-1} \bullet \mathbb{I}_{2^2} \bullet \dots \bullet \boldsymbol{\hbar}_2 \bullet \mathbb{I}_{2^{d-1}} \bullet \boldsymbol{\hbar}_1 \tag{4.73}$$

(cf. Definitions 2.6 and 2.8). Note that (4.73), item (I), Proposition 2.10, and Proposition 2.7 ensure that

$$\mathcal{R}(\boldsymbol{f}) = [\mathcal{R}(\boldsymbol{\hbar}_d)] \circ [\mathcal{R}(\boldsymbol{\hbar}_{d-1})] \circ \dots \circ [\mathcal{R}(\boldsymbol{\hbar}_1)] \in C(\mathbb{R}^{(2^d)}, \mathbb{R}).$$
(4.74)

Item (III), and induction therefore imply that for all  $x, y \in \mathbb{R}^{(2^d)}$  it holds that

$$\left| (\mathcal{R}(\mathbf{f}))(x) - (\mathcal{R}(\mathbf{f}))(y) \right| \le \left(\sqrt{32}\right)^d R^{(2^{d-1}+2^{d-2}+\ldots+2^0)} \|x-y\|_2$$
  
=  $2^{\frac{5d}{2}} R^{(2^d-1)} \|x-y\|_2.$  (4.75)

Next observe that the fact that for all  $i \in \{1, 2, ..., d\}, x, y \in [-R^{(2^{i-1})}, R^{(2^{i-1})}]$  it holds that  $xy \in [-R^{(2^i)}, R^{(2^i)}]$  demonstrates that for all  $i \in \{1, 2, ..., d-1\}$  it holds that  $p_i(D_i) \subseteq D_{i+1}$ . Combining this, (4.70), (4.71), (4.73), (4.84), item (II), and item (III) with Lemma 4.13 (applied with  $n \curvearrowleft d$ ,  $(d_0, d_1, ..., d_n) \curvearrowleft (2^d, 2^{d-1}, ..., 2^0)$ ,  $(L_i)_{i \in \{1, 2, ..., n\}} \curvearrowleft (\sqrt{32}R^{(2^{i-1})})_{i \in \{1, 2, ..., d\}}$ ,  $(\varepsilon_i)_{i \in \{1, 2, ..., n\}} \curvearrowleft (3R^2d^{\frac{1}{2}}2^{-2N_i-1})_{i \in \{1, 2, ..., d\}}, (D_1, D_2, ..., D_n) \curvearrowleft (D_1, D_2, ..., D_d), (f_1, f_2, ..., f_n) \curvearrowleft (p_1, p_2, ..., p_d), (g_1, g_2, ..., g_n) \curvearrowleft (\mathcal{R}(\mathscr{K}_1), \mathcal{R}(\mathscr{K}_2), ..., \mathcal{R}(\mathscr{K}_d))$  in the notation of Lemma 4.13) ensures that for all  $x = (x_1, x_2, ..., x_{2^d}) \in [-R, R]^{(2^d)}$  it holds that

$$\left| \begin{bmatrix} 2^{d} \\ \prod_{i=1}^{2^{d}} x_{i} \end{bmatrix} - (\mathcal{R}(\mathbf{f}))(x) \right| \\
= \left| (p_{d} \circ p_{d-1} \circ \ldots \circ p_{1})(x) - \left( [\mathcal{R}(\mathbf{\hbar}_{d})] \circ [\mathcal{R}(\mathbf{\hbar}_{d-1})] \circ \ldots \circ [\mathcal{R}(\mathbf{\hbar}_{1})] \right)(x) \right| \\
\leq \sum_{i=1}^{d} \left[ \left( \prod_{j=i+1}^{d} \sqrt{32R^{(2^{j-1})}} \right) 3R^{2} d^{\frac{1}{2}} 2^{-2N_{i}-1} \right] \\
= \left[ \sum_{i=1}^{d} 2^{\frac{5d-5i}{2}} R^{(2^{d}-2^{i})} 3R^{2} d^{\frac{1}{2}} 2^{-2N_{i}-1} \right] \leq \left[ \sum_{i=1}^{d} d^{-1} \varepsilon \right] = \varepsilon.$$
(4.76)

Furthermore, note that (4.73), item (V), and Lemma 2.11 demonstrate that

$$\mathbb{D}_{\mathcal{L}(\ell)-1}(\ell) = \mathbb{D}_{\mathcal{L}(\hbar_d)-1}(\hbar_d) = 2^{d-d+1} 3 = 6, \quad \text{and} \quad \mathbb{D}_1(\ell) = \mathbb{D}_1(\hbar_1) = 2^d 3. \quad (4.77)$$

Moreover, observe that (4.73), item (VII), and Proposition 2.18 show that

$$\mathcal{S}(\boldsymbol{\ell}) \leq \max\{\mathcal{S}(\boldsymbol{\hbar}_1), \mathcal{S}(\boldsymbol{\hbar}_2), \dots, \mathcal{S}(\boldsymbol{\hbar}_d)\} \leq 4.$$
(4.78)

In addition, note that the assumption that  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ , and  $R \in [1, \infty)$  ensure that for all  $i \in \{1, 2, \ldots, d\}$  it holds that

$$N_{i} = \left\lceil \frac{8d-5i}{4} + (2^{d-1} - 2^{i-1} + 1) \log_{2}(R) - \frac{1}{2} \log_{2}(\varepsilon) + \frac{1}{2} \right\rceil$$

$$\leq 2d + \frac{3}{2} + 2^{d-1} \lceil \log_{2}(R) \rceil - \frac{1}{2} \log_{2}(\varepsilon)$$

$$\leq 2^{d+1} + 2^{d-1} \lceil \log_{2}(R) \rceil - \frac{1}{2} \log_{2}(\varepsilon).$$
(4.79)

Thus, item (V) implies that for all  $i \in \{1, 2, ..., d\}$  it holds that

$$\begin{aligned} \mathcal{P}(\mathscr{R}_{i}) &= 2^{2d-2i}234 + 2^{d-i}49 + N_{i}(2^{2d-2i}144 + 2^{d-i}12) + \lceil \log_{2}(R) \rceil (2^{2d-2i}36 + 2^{d-i}6) \\ &\leq 2^{2d-2i}283 + 2^{2d-2i}156N_{i} + 2^{2d-2i}42\lceil \log_{2}(R) \rceil \\ &\leq 2^{2d-2i+9} + 2^{2d-2i+8}N_{i} + 2^{2d-2i+6}\lceil \log_{2}(R) \rceil \\ &\leq 2^{2d-2i+9} + 2^{2d-2i+8}(2^{d+1} + 2^{d-1}\lceil \log_{2}(R) \rceil - \frac{1}{2}\log_{2}(\varepsilon)) + 2^{2d-2i+6}\lceil \log_{2}(R) \rceil \\ &= 2^{2d-2i+9} + 2^{3d-2i+9} + (2^{3d-2i+7} + 2^{2d-2i+6})\lceil \log_{2}(R) \rceil - 2^{2d-2i+7}\log_{2}(\varepsilon) \\ &\leq 2^{3d-2i+10} + 2^{3d-2i+8}\lceil \log_{2}(R) \rceil - 2^{2d-2i+7}\log_{2}(\varepsilon) \\ &= 2^{2d-2i}(2^{d+10} + 2^{d+8}\lceil \log_{2}(R) \rceil - 2^{7}\log_{2}(\varepsilon)). \end{aligned}$$

Combining this and [3, Proposition 2.19] with the fact that  $\sum_{i=0}^{d-1} 4^i = \frac{4^d-1}{3} \leq \frac{2^{2d}}{3}$  shows that

$$\begin{aligned} \mathcal{P}(\boldsymbol{\ell}) &\leq 3 \left[ \sum_{i=1}^{d} \mathcal{P}(\boldsymbol{\aleph}_{i}) \right] - \mathcal{P}(\boldsymbol{\aleph}_{1}) - \mathcal{P}(\boldsymbol{\aleph}_{d}) \\ &\leq 3 \left[ \sum_{i=1}^{d} 2^{2d-2i} \left( 2^{d+10} + 2^{d+8} \lceil \log_{2}(R) \rceil - 2^{7} \log_{2}(\varepsilon) \right) \right] \\ &= 3 \left[ \sum_{i=0}^{d-1} 4^{i} \right] \left( 2^{d+10} + 2^{d+8} \lceil \log_{2}(R) \rceil - 2^{7} \log_{2}(\varepsilon) \right) \\ &\leq 2^{2d} \left( 2^{d+10} + 2^{d+8} \lceil \log_{2}(R) \rceil - 2^{7} \log_{2}(\varepsilon) \right) \\ &= 2^{3d+10} + 2^{3d+8} \lceil \log_{2}(R) \rceil - 2^{2d+7} \log_{2}(\varepsilon). \end{aligned}$$
(4.81)

Furthermore, observe that (4.73), (4.79), item (IV), and Proposition 2.10 show that

$$\begin{aligned} \mathcal{L}(\boldsymbol{\ell}) &= \left[\sum_{i=1}^{d} \mathcal{L}(\mathcal{R}_{i})\right] + \left[\sum_{i=1}^{d-1} \mathcal{L}(\mathbb{I}_{2^{i}})\right] - 2(d-1) \\ &= \left[\sum_{i=1}^{d} N_{i} + \left\lceil 2^{i-1} \log_{2}(R) \right\rceil + 7\right] + \left[\sum_{i=1}^{d-1} 2\right] - 2(d-1) \\ &\leq \left[\sum_{i=1}^{d} 2^{d+1} + 2^{d-1} \left\lceil \log_{2}(R) \right\rceil - \frac{1}{2} \log_{2}(\varepsilon) + 2^{i-1} \left\lceil \log_{2}(R) \right\rceil + 7\right] \\ &\leq \left[\sum_{i=1}^{d} 2^{d+1} + 2^{d} \left\lceil \log_{2}(R) \right\rceil - \frac{1}{2} \log_{2}(\varepsilon) + 7\right] \\ &= d2^{d+1} + d2^{d} \left\lceil \log_{2}(R) \right\rceil - \frac{d}{2} \log_{2}(\varepsilon) + 7d \\ &\leq d2^{d+2} + d2^{d} \left\lceil \log_{2}(R) \right\rceil - \frac{d}{2} \log_{2}(\varepsilon). \end{aligned}$$
(4.82)

Combining this with (4.74), (4.75), (4.76), (4.77), (4.78), and (4.81) establishes items (i), (ii), (iii), (iv), (v), (v), and (vii). The proof of Lemma 4.14 is thus complete.

**Lemma 4.15.** Let  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $R \in (1,\infty)$ ,  $\gamma \in (0,1]$ ,  $\beta \in [1,\infty)$ . Then there exists  $\ell \in \mathbb{N}$  such that

(i) it holds that  $\mathcal{R}(\mathscr{L}) \in C(\mathbb{R}^d, \mathbb{R})$ ,

(*ii*) it holds that  $\sup_{x=(x_1,\dots,x_d)\in [-R,R]^d} |\gamma\beta^d \prod_{i=1}^d x_i - (\mathcal{R}(\boldsymbol{\ell}))(x)| \leq \varepsilon$ ,

(iii) it holds for all  $x, y \in \mathbb{R}^d$  that  $\left| (\mathcal{R}(\mathbf{f}))(x) - (\mathcal{R}(\mathbf{f}))(y) \right| \le \sqrt{32} d^{\frac{5}{2}} \beta^d R^{2d-1} ||x-y||_2$ ,

- (iv) it holds that  $\mathcal{L}(\mathbf{f}) \leq 8d^2 + 2d^2 \lceil \log_2(R) \rceil + d \log_2(\varepsilon^{-1}) + d^2 \lceil \log_2(\beta) \rceil + 2$ ,
- (v) it holds that that  $\mathbb{D}_1(\mathcal{L}) \leq 2d$  and  $\mathbb{D}_{\mathcal{H}(\mathcal{L})}(\mathcal{L}) = 2$ ,
- (vi) it holds that  $\mathcal{P}(\mathbf{f}) \leq 8203d^3 + 2048d^3 \lceil \log_2(R) \rceil 512d^2 \log_2(\varepsilon) + 514d^3 \log_2(\beta)$ , and
- (vii) it holds that  $\mathcal{S}(\not e) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, 3.14, and 4.8).

Proof of Lemma 4.15. Throughout this proof assume w.l.o.g. that d > 1 (cf. Corollary 4.6), let  $D \in \mathbb{N}$  satisfy  $D = 2^{\lceil \log_2(d) \rceil}$ , let  $A \in \mathbb{R}^{D \times d}$ ,  $B \in \mathbb{R}^D$  satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$Ax + B = (\gamma x_1, x_2, \dots, x_d, 1, 1, \dots, 1), \tag{4.83}$$

and let  $q_1 \in (\mathbb{R}^{D \times d} \times \mathbb{R}^D) \subseteq \mathbf{N}$  satisfy  $q_1 = (A, B)$  (cf. Definitions 2.1 and 4.8). Note that Lemma 4.14 (applied with  $d \curvearrowright \lceil \log_2(d) \rceil$ ,  $R \curvearrowleft R$ ,  $\varepsilon \curvearrowleft \varepsilon \beta^{-d}$  in the notation of Lemma 4.14) ensures that there exists  $q_2 \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(g_2) \in C(\mathbb{R}^D, \mathbb{R})$ ,
- (II) it holds that  $\sup_{x=(x_1,\dots,x_D)\in [-R,R]^D} \left|\prod_{i=1}^D x_i (\mathcal{R}(\mathbf{f}))(x)\right| \leq \varepsilon \beta^{-d}$ ,
- (III) it holds for all  $x, y \in \mathbb{R}^D$  that  $\left| (\mathcal{R}(g_2))(x) (\mathcal{R}(g_2))(y) \right| \le D^{\frac{5}{2}} R^{D-1} ||x y||_2$ ,

(IV) it holds that  $\mathcal{L}(g_2) \leq 4\lceil \log_2(d) \rceil D + \lceil \log_2(d) \rceil D \lceil \log_2(R) \rceil + \frac{\lceil \log_2(d) \rceil (d \log_2(\beta) - \log_2(\varepsilon))}{2}$ ,

(V) it holds that that  $\mathbb{D}_1(\mathfrak{g}_2) = 3D$  and  $\mathbb{D}_{\mathcal{H}(\mathfrak{g}_2)}(\mathfrak{g}_2) = 6$ ,

(VI) it holds that  $\mathcal{P}(g_2) \leq 2^{10}D^3 + 2^8D^3 \lceil \log_2(R) \rceil + 2^7D^2(d \log_2(\beta) - \log_2(\varepsilon))$ , and

(VII) it holds that  $\mathcal{S}(g_2) \leq 4$ 

(cf. Definitions 2.3, 2.13, and 3.14). Observe that Corollary 4.6 (applied with  $\beta \curvearrowleft \beta^d$ ,  $L \curvearrowleft d\lceil \log_2(\beta) \rceil$  in the notation of Corollary 4.6) shows that there exists  $g_3 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_3))(x) = \beta^d x$ ,
- (B) it holds that  $\mathcal{D}(g_3) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{d \lceil \log_2(\beta) \rceil + 2}$ ,
- (C) it holds that  $\mathbb{S}_0(\mathfrak{f}) \leq 1$ ,  $\mathbb{S}_1(\mathfrak{f}) \leq 2$ , and  $\mathcal{S}(\mathfrak{g}_3) \leq 2$ , and
- (D) it holds that  $\mathcal{P}(\mathfrak{Q}_3) \leq 6d \lceil \log_2(\beta) \rceil + 1$

Note that Proposition 2.10, item (I), item (B), and the fact that  $\mathcal{R}(g_1) \in C(\mathbb{R}^d, \mathbb{R}^D)$  imply that

$$\mathcal{R}(\boldsymbol{g}_3 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_2 \bullet \mathbb{I}_D \bullet \boldsymbol{g}_1) = [\mathcal{R}(\boldsymbol{g}_3)] \circ [\mathcal{R}(\boldsymbol{g}_2)] \circ [\mathcal{R}(\boldsymbol{g}_1)] \in C(\mathbb{R}^d, \mathbb{R})$$
(4.84)

(cf. Definitions 2.6 and 2.8). Furthermore, observe that item (III), item (A), (4.83), the fact that  $D \leq 2d$ , and the assumption that R > 1 show that for all  $x, y \in \mathbb{R}^d$  it holds that

$$\begin{aligned} \left| (\mathcal{R}(\mathcal{Q}_{3} \bullet \mathbb{I}_{1} \bullet \mathcal{Q}_{2} \bullet \mathbb{I}_{D} \bullet \mathcal{Q}_{1}))(x) - (\mathcal{R}(\mathcal{Q}_{3} \bullet \mathbb{I}_{1} \bullet \mathcal{Q}_{2} \bullet \mathbb{I}_{D} \bullet \mathcal{Q}_{1}))(y) \right| \\ &= \left| ([\mathcal{R}(\mathcal{Q}_{3})] \circ [\mathcal{R}(\mathcal{Q}_{2})] \circ [\mathcal{R}(\mathcal{Q}_{1})])(x) - ([\mathcal{R}(\mathcal{Q}_{3})] \circ [\mathcal{R}(\mathcal{Q}_{2})] \circ [\mathcal{R}(\mathcal{Q}_{1})])(y) \right| \\ &\leq \beta^{d} D^{\frac{5}{2}} R^{D-1} \| (\mathcal{R}(\mathcal{Q}_{1}))(x) - (\mathcal{R}(\mathcal{Q}_{1}))(y) \|_{2} \\ &= \beta^{d} D^{\frac{5}{2}} R^{D-1} \| Ax + B - (Ay + B) \|_{2} \\ &= D^{\frac{5}{2}} \beta^{d} R^{D-1} \| x - y \|_{2} \leq \sqrt{32} d^{\frac{5}{2}} \beta^{d} R^{2d-1} \| x - y \|_{2}. \end{aligned}$$
 (4.85)

Moreover, note that (4.83) and the assumption that  $R > 1 \ge \gamma$  ensure that for all  $x \in [-R, R]^d$  it holds that  $Ax + B \in [-R, R]^D$ . Item (II), item (A), (4.83), and (4.84) therefore demonstrate that for all  $x = (x_1, \ldots, x_d) \in [-R, R]^d$  it holds that

$$\left| \left[ \gamma \beta^{d} \prod_{i=1}^{d} x_{i} \right] - \left( \mathcal{R}(\mathcal{g}_{3} \bullet \mathbb{I}_{1} \bullet \mathcal{g}_{2} \bullet \mathbb{I}_{D} \bullet \mathcal{g}_{1}) \right)(x) \right|$$

$$= \left| \left[ \gamma \beta^{d} \prod_{i=1}^{d} x_{i} \right] - \left( \left[ \mathcal{R}(\mathcal{g}_{3}) \right] \circ \left[ \mathcal{R}(\mathcal{g}_{2}) \right] \circ \left[ \mathcal{R}(\mathcal{g}_{1}) \right] \right)(x) \right|$$

$$= \beta^{d} \left| \left[ 1^{D-d} \gamma \prod_{i=1}^{d} x_{i} \right] - \left( \mathcal{R}(\mathcal{g}_{2}) \right)(\gamma x_{1}, x_{2}, \dots, x_{d}, 1, 1, \dots, 1) \right| \leq \beta^{d} \varepsilon \beta^{-d} = \varepsilon.$$

$$(4.86)$$

In addition, observe that (4.83), item (VII), item (C), and Proposition 2.17 imply that

$$\mathcal{S}(\boldsymbol{g}_3 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_2 \bullet \mathbb{I}_D \bullet \boldsymbol{g}_1) = \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \le \max\{2, 4, 1\} = 4.$$
(4.87)

Furthermore, note that item (IV), (4.89), Proposition 2.10, Proposition 2.7, and the fact that  $\max\{D, 2\lceil \log_2(d)\rceil\} \le 2d$  imply that

$$\begin{aligned} \mathcal{L}(\boldsymbol{g}_{3} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{2} \bullet \mathbb{I}_{D} \bullet \boldsymbol{g}_{1}) \\ &= \mathcal{L}(\boldsymbol{g}_{3}) + \mathcal{L}(\boldsymbol{g}_{2}) + \mathcal{L}(\boldsymbol{g}_{1}) + 4 - 4 \\ &= \mathcal{L}(\boldsymbol{g}_{3}) + \mathcal{L}(\boldsymbol{g}_{2}) + 1 \end{aligned}$$

$$\leq (d \lceil \log_{2}(\beta) \rceil + 1) + \left(4 \lceil \log_{2}(d) \rceil D + \lceil \log_{2}(d) \rceil D \lceil \log_{2}(R) \rceil + \frac{\lceil \log_{2}(d) \rceil (d \log_{2}(\beta) - \log_{2}(\varepsilon))}{2} \right) + 1 \\ &\leq 8d^{2} + 2d^{2} \lceil \log_{2}(R) \rceil + \log_{2}(\varepsilon^{-1})d + d^{2} \lceil \log_{2}(\beta) \rceil + 2 \end{aligned}$$

$$(4.88)$$

This, Lemma 2.11, item (V), item (B), and (4.83) imply that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g_3) + \mathcal{L}(g_2) + 1]$  it holds that

$$\mathbb{D}_{k}(\boldsymbol{g}_{3} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{2} \bullet \mathbb{I}_{D} \bullet \boldsymbol{g}_{1}) = \begin{cases} d & : k = 0\\ 2D & : k = 1\\ \mathbb{D}_{k-1}(\boldsymbol{g}_{2}) & : k \in \mathbb{N} \cap (1, \mathcal{L}(\boldsymbol{g}_{2})] \\ 2 & : k \in \mathbb{N} \cap (\mathcal{L}(\boldsymbol{g}_{2}), \mathcal{L}(\boldsymbol{g}_{3}) + \mathcal{L}(\boldsymbol{g}_{2}) + 1)\\ 1 & : k = \mathcal{L}(\boldsymbol{g}_{3}) + \mathcal{L}(\boldsymbol{g}_{2}) + 1. \end{cases}$$
(4.89)

Combining this, item (V), item (VI), and item (B) with the fact that  $\mathbb{D}_0(\mathfrak{g}_2) = D$ , and  $\max\{4, D\} \leq 2d$  shows that

$$\begin{aligned} \mathcal{P}(\boldsymbol{g}_{3} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{2} \bullet \mathbb{I}_{D} \bullet \boldsymbol{g}_{1}) \\ &= \sum_{k=1}^{\mathcal{L}(\boldsymbol{g}_{3})+\mathcal{L}(\boldsymbol{g}_{2})+1} \mathbb{D}_{k}(\boldsymbol{g}_{3} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{2} \bullet \mathbb{I}_{D} \bullet \boldsymbol{g}_{1})(\mathbb{D}_{k-1}(\boldsymbol{g}_{3} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{2} \bullet \mathbb{I}_{D} \bullet \boldsymbol{g}_{1}) + 1) \\ &= 2D(d+1) + \mathbb{D}_{1}(\boldsymbol{g}_{2})(2\mathbb{D}_{0}(\boldsymbol{g}_{2}) + 1) + \left[\sum_{k=2}^{\mathcal{L}(\boldsymbol{g}_{2})-1} \mathbb{D}_{k}(\boldsymbol{g}_{2})(\mathbb{D}_{k-1}(\boldsymbol{g}_{2}) + 1)\right] \\ &+ 2\mathbb{D}_{0}(\boldsymbol{g}_{3})(\mathbb{D}_{\mathcal{L}(\boldsymbol{g}_{2})-1}(\boldsymbol{g}_{2}) + 1) + \left[\sum_{k=2}^{\mathcal{L}(\boldsymbol{g}_{3})} \mathbb{D}_{k}(\boldsymbol{g}_{3})(\mathbb{D}_{k-1}(\boldsymbol{g}_{3}) + 1)\right] \\ &= 2D(d+1) + \mathbb{D}_{1}(\boldsymbol{g}_{2})\mathbb{D}_{0}(\boldsymbol{g}_{2}) + \left[\sum_{k=1}^{\mathcal{L}(\boldsymbol{g}_{2})} \mathbb{D}_{k}(\boldsymbol{g}_{2})(\mathbb{D}_{k-1}(\boldsymbol{g}_{2}) + 1)\right] \\ &+ \left(\mathbb{D}_{\mathcal{L}(\boldsymbol{g}_{2})-1}(\boldsymbol{g}_{2}) + 1\right) - 3 + \left[\sum_{k=1}^{\mathcal{L}(\boldsymbol{g}_{3})} \mathbb{D}_{k}(\boldsymbol{g}_{3})(\mathbb{D}_{k-1}(\boldsymbol{g}_{3}) + 1)\right] \\ &= 2D(d+1) + 3D^{2} + \mathcal{P}(\boldsymbol{g}_{2}) + 7 - 3 + \mathcal{P}(\boldsymbol{g}_{3}) \\ &\leq 2D(d+1) + 3D^{2} + 2^{10}D^{3} + 2^{8}D^{3}\lceil\log_{2}(R)\rceil + 2^{7}D^{2}(d\log_{2}(\beta) - \log_{2}(\varepsilon)) \\ &+ 4 + 6d\lceil\log_{2}(\beta)\rceil + 1 \\ &\leq 4d(d+1) + 12d^{2} + 2^{13}d^{3} + 2^{11}d^{3}\lceil\log_{2}(R)\rceil - 2^{9}d^{2}\log_{2}(\varepsilon) + (2^{9}d^{3} + 6d)\log_{2}(\beta) \\ &+ 6d + 5 \\ &\leq (2+1+6+2^{13}+2)d^{3} + 2^{11}d^{3}\lceil\log_{2}(R)\rceil - 2^{9}d^{2}\log_{2}(\varepsilon) + (2^{9}+2)d^{3}\log_{2}(\beta) \\ &= 8203d^{3} + 2048d^{3}\lceil\log_{2}(R)\rceil - 512d^{2}\log_{2}(\varepsilon) + 514d^{3}\log_{2}(\beta). \end{aligned}$$

This, (4.84), (4.85), (4.86), (4.87), (4.89), and (4.90) establish items (i), (ii), (iii), (iv), (v), (vi), and (vii). The proof of Lemma 4.15 is thus complete.

**Corollary 4.16.** Let  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $\gamma \in (0, 1]$ ,  $\beta \in [1, \infty)$ . Then there exists  $\not \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\not e) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (ii) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} |\gamma\beta^d\prod_{i=1}^d x_i (\mathcal{R}(\mathscr{I}))(x)| \leq \varepsilon$ ,
- (iii) it holds that  $\mathcal{L}(\mathscr{L}) \leq 59d^2 \max\{1, \lceil \log_2(|a|) \rceil, \lceil \log_2(|b|) \rceil, \log_2(\varepsilon^{-1}), \lceil \log_2(\beta) \rceil\},\$
- (iv) it holds that  $\mathcal{P}(\mathcal{L}) \leq 12143d^3 \max\{1, \lceil \log_2(|a|) \rceil, \lceil \log_2(|b|) \rceil, \log_2(\varepsilon^{-1}), \lceil \log_2(\beta) \rceil\}$ , and
- (v) it holds that  $\mathcal{S}(\mathbf{p}) \leq 1$
- (cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Corollary 4.16. Throughout this proof assume w.l.o.g. that  $\max\{|a|, |b|\} > 1 > \varepsilon$  let  $R \in (1, \infty)$  satisfy  $R = \max\{|a|, |b|\}$ . Observe that Lemma 4.15 (applied with  $d \curvearrowleft d, \varepsilon \backsim \varepsilon$ ,  $R \curvearrowleft R, \gamma \backsim \gamma, \beta \backsim \beta$  in the notation of Lemma 4.15) shows that there exists  $g \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(q) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (II) it holds that  $\sup_{x=(x_1,\dots,x_d)\in [-R,R]^d} |\gamma\beta^d\prod_{i=1}^d x_i (\mathcal{R}(g))(x)| \le \varepsilon$ ,
- (III) it holds that  $\mathcal{L}(q) = 8d^2 + 2d^2 \lceil \log_2(R) \rceil + \log_2(\varepsilon^{-1})d + d^2 \lceil \log_2(\beta) \rceil + 2$ ,
- (IV) it holds that that  $\mathbb{D}_{\mathcal{H}(g)}(g) = 2$ ,
- (V) it holds that  $\mathcal{P}(g) \leq 8203d^3 + 2048d^3 \lceil \log_2(R) \rceil 512d^2 \log_2(\varepsilon) + 514d^3 \log_2(\beta)$ , and
- (VI) it holds that  $\mathcal{S}(q) \leq 4$

Note that item (III) implies that

$$\mathcal{L}(q) = 8d^{2} + 2d^{2} \lceil \log_{2}(R) \rceil + d \log_{2}(\varepsilon^{-1}) + d^{2} \lceil \log_{2}(\beta) \rceil + 2$$
  

$$\leq (8 + 2 + 1 + 1 + 2)d^{2} \max\{1, \lceil \log_{2}(R) \rceil, \log_{2}(\varepsilon^{-1}), \lceil \log_{2}(\beta) \rceil\}$$

$$= 14d^{2} \max\{\lceil \log_{2}(R) \rceil, \log_{2}(\varepsilon^{-1}), \lceil \log_{2}(\beta) \rceil\}.$$
(4.91)

Observe that item (IV) demonstrates that

$$\mathcal{P}(\boldsymbol{g}) \leq 8203d^{3} + 2048d^{3}\lceil \log_{2}(R) \rceil - 512d^{2}\log_{2}(\varepsilon) + 514d^{3}\log_{2}(\beta)$$
  
$$\leq (8203 + 2048 + 512 + 514)d^{3}\max\{1, \lceil \log_{2}(R) \rceil, \log_{2}(\varepsilon^{-1}), \lceil \log_{2}(\beta) \rceil\}$$
(4.92)  
$$= 11277d^{3}\max\{\lceil \log_{2}(R) \rceil, \log_{2}(\varepsilon^{-1}), \lceil \log_{2}(\beta) \rceil\}.$$

Combining this and (4.91) with Corollary 4.4 (applied with  $\not \sim g$ ,  $d \sim d$  in the notation of Corollary 4.4) shows that there exists  $\not \in \mathbf{N}$  which satisfies that

- (A) it holds that  $\mathcal{R}(\mathcal{L}) = \mathcal{R}(\mathcal{Q}) \in C(\mathbb{R}^d, \mathbb{R}),$
- (B) it holds that

$$\mathcal{L}(\boldsymbol{\ell}) = 4(14d^2 \max\{\lceil \log_2(R) \rceil, \log_2(\varepsilon^{-1}), \lceil \log_2(\beta) \rceil\}) + 3$$
  
$$\leq 59d^2 \max\{\lceil \log_2(R) \rceil, \log_2(\varepsilon^{-1}), \lceil \log_2(\beta) \rceil\},$$
(4.93)

(C) it holds that

$$\mathcal{P}(\boldsymbol{\ell}) \le (11277d^3 + 2 + 840d^2 + 24) \max\{\lceil \log_2(R) \rceil, \log_2(\varepsilon^{-1}), \lceil \log_2(\beta) \rceil\}$$
  
= 12143d<sup>3</sup> max{[log<sub>2</sub>(R)], log<sub>2</sub>(\varepsilon^{-1}), [log<sub>2</sub>(\varepsilon)]}, (4.94)

and

(D) it holds that  $\mathcal{S}(\not e) \leq 1$ .

Note that item (II) and item (A) prove that for all  $x = (x_1, \ldots, x_d) \in [a, b]^d \subseteq [-R, R]^d$  it holds that

$$\left|\gamma\beta^{d}\prod_{i=1}^{d}x_{i}-(\mathcal{R}(\boldsymbol{f}))(x)\right|\leq\varepsilon.$$
(4.95)

Combining this with items (A), (B), (C), and (D) establishes items (I), (II), (III), (IV), and (V). The proof of Corollary 4.16 is thus complete.  $\Box$ 

## 4.4 Upper bounds for approximations of periodic functions

**Lemma 4.17.** Let  $\lambda \in (0, \infty)$  and let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}$ ,  $x \in [2k - 1, 2k + 1)$  that  $\mathfrak{s}(x) = 1 - |x - 2k|$ . Then it holds for all  $x \in \mathbb{R}$  that

$$\mathfrak{s}(2\lambda x) = 2\mathfrak{s}(\lambda x) - 4\mathfrak{R}(\mathfrak{s}(\lambda x) - \frac{1}{2}) \tag{4.96}$$

(cf. Definition 2.2).

Proof of Lemma 4.17. Observe that the fact that for all  $k \in \mathbb{Z}$ ,  $x \in [2k, 2k + 1)$  it holds that  $\mathfrak{s}(x) = x - 2k$  shows that for all  $k \in \mathbb{Z}$ ,  $x \in \mathbb{R}$  with  $2\lambda x \in [4k, 4k + 1)$  it holds that

$$2\mathfrak{s}(\lambda x) - 4\mathfrak{R}\big(\mathfrak{s}(\lambda x) - \frac{1}{2}\big) = 2(\lambda x - 2k) - 4\mathfrak{R}\big((\lambda x - 2k) - \frac{1}{2}\big)$$
$$= (2\lambda x - 4k) - 2\mathfrak{R}\big(2\lambda x - 4k - 1\big)$$
$$= 2\lambda x - 4k$$
$$= \mathfrak{s}(2\lambda x)$$
(4.97)

(cf. Definition 2.2). Furthermore, note that the fact that for all  $k \in \mathbb{Z}$ ,  $x \in [2k, 2k + 1)$ ,  $y \in [2k + 1, 2k + 2)$  it holds that  $\mathfrak{s}(x) = x - 2k$  and  $\mathfrak{s}(y) = -y + 2k + 2$  demonstrates that for all  $k \in \mathbb{Z}$ ,  $x \in \mathbb{R}$  with  $2\lambda x \in [4k + 1, 4k + 2)$  it holds that

$$2\mathfrak{s}(\lambda x) - 4\mathfrak{R}(\mathfrak{s}(\lambda x) - \frac{1}{2}) = 2(\lambda x - 2k) - 4\mathfrak{R}((\lambda x - 2k) - \frac{1}{2})$$
  
$$= (2\lambda x - 4k) - 2\mathfrak{R}(2\lambda x - 4k - 1)$$
  
$$= -2\lambda x + (4k + 2)$$
  
$$= \mathfrak{s}(2\lambda x).$$
  
(4.98)

Moreover, observe that the fact that for all  $k \in \mathbb{Z}$ ,  $x \in [2k, 2k+1)$ ,  $y \in [2k+1, 2k+2)$  it holds that  $\mathfrak{s}(x) = x - 2k$  and  $\mathfrak{s}(y) = -y + 2k + 2$  ensures that for all  $k \in \mathbb{Z}$ ,  $x \in \mathbb{R}$  with  $2\lambda x \in [4k+2, 4k+3)$  it holds that

$$2\mathfrak{s}(\lambda x) - 4\mathfrak{R}(\mathfrak{s}(\lambda x) - \frac{1}{2}) = 2(-\lambda x + 2k + 2) - 4\mathfrak{R}((-\lambda x + 2k + 2) - \frac{1}{2}) = -2\lambda x + 4k + 4 - 2\mathfrak{R}(-2\lambda x + 4k + 3) = 2\lambda x - (4k + 2) = \mathfrak{s}(2\lambda x).$$
(4.99)

In addition, note that the fact that for all  $k \in \mathbb{Z}$ ,  $y \in [2k+1, 2k+2)$  it holds that  $\mathfrak{s}(y) = -y + 2k + 2$  implies that for all  $k \in \mathbb{Z}$ ,  $x \in \mathbb{R}$  with  $2\lambda x \in [4k+3, 4k+4)$  it holds that

$$2\mathfrak{s}(\lambda x) - 4\mathfrak{R}(\mathfrak{s}(\lambda x) - \frac{1}{2}) = 2(-\lambda x + 2k + 2) - 4\mathfrak{R}((-\lambda x + 2k + 2) - \frac{1}{2}) = -2\lambda x + 4k + 4 - 2\mathfrak{R}(-2\lambda x + 4k + 3)$$
(4.100)  
=  $-2\lambda x + (4k + 4) = \mathfrak{s}(2\lambda x).$ 

Combining this with (4.97), (4.98), and (4.99) ensures (4.96). The proof of Lemma 4.17 is thus complete.

**Lemma 4.18.** Let  $B \in (0, \infty)$ , let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}$ ,  $x \in [2k - 1, 2k + 1)$  that  $\mathfrak{s}(x) = 1 - |x - 2k|$ , let  $g \in ((\mathbb{R}^{4 \times 1} \times \mathbb{R}^4) \times (\mathbb{R}^{1 \times 4} \times \mathbb{R}^1)) \subseteq \mathbb{N}$  satisfy

$$g = \left( \left( \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\-1\\-1 \end{pmatrix} \right), \left( \begin{pmatrix} 1 & 1 & -2 & -2 \end{pmatrix}, 0 \right) \right),$$
(4.101)

and let  $\not e_n \in \left( (\mathbb{R}^{4 \times 1} \times \mathbb{R}^4) \times (\mathbb{R}^{1 \times 4} \times \mathbb{R}^1) \right) \subseteq \mathbf{N}, n \in \mathbb{N}_0, \text{ satisfy for all } n \in \mathbb{N} \text{ that}$ 

$$\boldsymbol{\pounds}_{0} = \left( \left( \begin{pmatrix} 2B^{-1} \\ 2B^{-1} \\ -2B^{-1} \\ -2B^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right), \left( \begin{pmatrix} 1 & -2 & 1 & -2 \end{pmatrix}, 0 \right) \right) \quad and \quad \boldsymbol{\pounds}_{n} = \boldsymbol{g} \bullet \boldsymbol{\pounds}_{n-1} \quad (4.102)$$

(cf. Definitions 2.1 and 2.8). Then

- (i) it holds for all  $n \in \mathbb{N}_0$ ,  $x \in [0, B]$  that  $(\mathcal{R}(\mathscr{J}_n))(-x) = (\mathcal{R}(\mathscr{J}_n))(x) = \mathfrak{s}(2^{n+1}B^{-1}x)$ ,
- (ii) it holds for all  $n \in \mathbb{N}$ ,  $x \in (B, \infty)$  that  $(\mathcal{R}(\mathcal{f}_n))(-x) = (\mathcal{R}(\mathcal{f}_n))(x) = 0$ ,
- (iii) it holds for all  $n \in \mathbb{N}_0$  that  $\mathcal{D}(\mathcal{L}_n) = (1, 4, 4, \dots, 4, 1) \in \mathbb{N}^{\mathcal{L}(\mathcal{L}_n)+1}$
- (iv) it holds for all  $n \in \mathbb{N}_0$  that  $\mathcal{L}(\mathbb{A}_n) = n+2$ , and
- (v) it holds for all  $n \in \mathbb{N}_0$  that  $S(\mathbb{A}_n) \leq \max\{2B^{-1}, 2\}, \ \mathbb{S}_0(\mathbb{A}_n) = \max\{2B^{-1}, 1\}, \ and \ \mathbb{S}_1(\mathbb{A}_n) \leq 2$

(cf. Definitions 2.3 and 2.13).

Proof of Lemma 4.18. Throughout this proof let  $W, \mathfrak{W} \in \mathbb{R}^{4 \times 4}$ , satisfy

$$W = \mathcal{W}_{1,g}\mathcal{W}_{2,g}$$
 and  $\mathfrak{W} = \mathcal{W}_{1,g}\mathcal{W}_{2,\ell_0}$ , (4.103)

Observe that (4.101), (4.102), and Proposition 2.10 demonstrate that for all  $n \in \mathbb{N}$  it holds that

$$\mathcal{L}(\boldsymbol{f}_0) = 2 \quad \text{and} \quad \mathcal{L}(\boldsymbol{f}_n) = \mathcal{L}(\boldsymbol{g} \bullet \boldsymbol{f}_{n-1}) = \mathcal{L}(\boldsymbol{g}) + \mathcal{L}(\boldsymbol{f}_{n-1}) - 1 = \mathcal{L}(\boldsymbol{f}_{n-1}) + 1. \quad (4.104)$$

Hence induction establishes that for all  $n \in \mathbb{N}_0$  it holds that

$$\mathcal{L}(\boldsymbol{\rho}_n) = n+2. \tag{4.105}$$

This establishes item (iv). Furthermore, note that (2.10), (4.101), (4.102), (4.103) and the fact that  $\mathcal{W}_{1,g}\mathcal{B}_{2,\ell_0} + \mathcal{B}_{1,g} = \mathcal{B}_{1,g}$  demonstrate that

$$\boldsymbol{f}_1 = \boldsymbol{g} \bullet \boldsymbol{f}_0 = \left( (\mathcal{W}_{1,\boldsymbol{f}_0}, \mathcal{B}_{1,\boldsymbol{f}_0}), (\mathfrak{W}, \mathcal{B}_{1,\boldsymbol{g}}), (\mathcal{W}_{2,\boldsymbol{g}}, \mathcal{B}_{2,\boldsymbol{g}}) \right).$$
(4.106)

This, (2.10), (4.101), (4.102), (4.103) and the fact that  $\mathcal{W}_{1,g}\mathcal{B}_{2,g} + \mathcal{B}_{1,g} = \mathcal{B}_{1,g}$  demonstrate that

$$\boldsymbol{f}_{2} = \boldsymbol{g} \bullet \boldsymbol{f}_{1} = \big( (\mathcal{W}_{1,\boldsymbol{f}_{0}}, \mathcal{B}_{1,\boldsymbol{f}_{0}}), (\mathfrak{W}, \mathcal{B}_{1,\boldsymbol{g}}), (W, \mathcal{B}_{1,\boldsymbol{g}}), (\mathcal{W}_{2,\boldsymbol{g}}, \mathcal{B}_{2,\boldsymbol{g}}) \big).$$
(4.107)

Combining this, (2.10), (4.105), and (4.103) with induction demonstrates that for all  $n \in \mathbb{N} \cap [2, \infty)$  it holds that

$$\boldsymbol{\ell}_{n} = ((\mathcal{W}_{1,\boldsymbol{\ell}_{0}},\mathcal{B}_{1,\boldsymbol{\ell}_{0}}),(\mathfrak{W},\mathcal{B}_{1,\boldsymbol{g}}),(W,\mathcal{B}_{1,\boldsymbol{g}}),(W,\mathcal{B}_{1,\boldsymbol{g}}),\ldots,(W,\mathcal{B}_{1,\boldsymbol{g}}),(\mathcal{W}_{2,\boldsymbol{g}},\mathcal{B}_{2,\boldsymbol{g}})) \\ \in \left((\mathbb{R}^{4\times1}\times\mathbb{R}^{4})\times\left(\times_{k=1}^{n}(\mathbb{R}^{4\times4}\times\mathbb{R}^{4})\right)\times(\mathbb{R}^{1\times4}\times\mathbb{R}^{1})\right).$$

$$(4.108)$$

This, (4.101), (4.105), (4.106), (4.107), and (4.108) show that for all  $n \in \mathbb{N}_0$ ,  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{L}_n)]$  it holds that

$$\mathbb{D}_{k}(\mathscr{f}_{n}) = \begin{cases} 1 & : k \in \{0, \mathcal{L}(\mathscr{f}_{n})\} \\ 4 & : k \in \mathbb{N} \cap (0, \mathcal{L}(\mathscr{f}_{n})). \end{cases}$$
(4.109)

This establishes item (iii). Moreover, observe that (4.101) and the fact that for all  $x \in \mathbb{R}$  it holds that  $\Re(|x|) = \Re(x) + \Re(-x)$  and  $\Re(|x|-1) = \Re(x-1) + \Re(-x-1)$  show that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathcal{F}_0))(x) = \Re(2B^{-1}|x|+0) - 2\Re(2B^{-1}|x|-1) = \begin{cases} 2B^{-1}|x| & : 0 \le |x| \le \frac{B}{2} \\ 2 - 2B^{-1}|x| & : |x| \ge \frac{B}{2} \end{cases}$$
(4.110)

(cf. Definitions 2.2 and 2.3). This ensures that for all  $x \in [0, \frac{B}{2}], y \in [\frac{B}{2}, B], z \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathbf{f}_0))(x) = \mathfrak{s}(2B^{-1}x), \quad (\mathcal{R}(\mathbf{f}_0))(y) = \mathfrak{s}(2B^{-1}y), \quad \text{and} \quad (\mathcal{R}(\mathbf{f}_0))(-z) = (\mathcal{R}(\mathbf{f}_0))(z)$$
(4.111)

. In addition, note that (4.101), (4.102), and Proposition 2.10 imply that for all  $n \in \mathbb{N}, x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\boldsymbol{f}_n))(x) = (\mathcal{R}(\boldsymbol{g} \bullet \boldsymbol{f}_{n-1}))(x) = (\mathcal{R}(\boldsymbol{g}))((\mathcal{R}(\boldsymbol{f}_{n-1}))(x)) \quad \text{and} \quad (\mathcal{R}(\boldsymbol{g}))(0) = 0.$$
(4.112)

This shows that for all  $n \in \mathbb{N}$ ,  $x \in [0, \infty)$  with  $(\mathcal{R}(\mathcal{L}_{n-1}))(-x) = (\mathcal{R}(\mathcal{L}_{n-1}))(x)$  it holds that

$$(\mathcal{R}(\mathcal{f}_n))(-x) = (\mathcal{R}(\mathcal{g}))((\mathcal{R}(\mathcal{f}_{n-1}))(-x)) = (\mathcal{R}(\mathcal{g}))((\mathcal{R}(\mathcal{f}_{n-1}))(x)) = (\mathcal{R}(\mathcal{f}_n))(x).$$
(4.113)

Furthermore, observe that (4.101), (4.112), Proposition 2.10, and Lemma 4.17 (applied with  $\lambda \curvearrowleft 2^n B^{-1}$  for  $n \in \mathbb{N}$  in the notation of Lemma 4.17) ensure that for all  $x \in [0, B]$ ,  $n \in \mathbb{N}$  with  $(\mathcal{R}(\ell_{n-1}))(x) = \mathfrak{s}(2^n B^{-1} x)$  it holds that

$$(\mathcal{R}(\mathcal{L}_{n}))(x) = (\mathcal{R}(\mathcal{Q}))((\mathcal{R}(\mathcal{L}_{n-1}))(x)) = (\mathcal{R}(\mathcal{Q}))(\mathfrak{s}(2^{n}B^{-1}x)) = 2\mathfrak{R}(\mathfrak{s}(2^{n}B^{-1}x)) - 4\mathfrak{R}(\mathfrak{s}(2^{n}B^{-1}x) - \frac{1}{2}) = 2\mathfrak{s}(2^{n}B^{-1}x) - 4\mathfrak{R}(\mathfrak{s}(2^{n}B^{-1}x) - \frac{1}{2}) = \mathfrak{s}(2^{n+1}B^{-1}x).$$
(4.114)

Combining this, (4.111), and (4.113) with induction establishes item (i). Moreover, note that (4.110) and (4.112) demonstrate that for all  $x \in (B, \infty)$  it holds that

$$(\mathcal{R}(\mathcal{J}_1))(x) = (\mathcal{R}(\mathcal{Q}))((\mathcal{R}(\mathcal{J}_0))(x)) = (\mathcal{R}(\mathcal{Q}))(2 - 2B^{-1}x)$$
  
=  $2\mathfrak{R}(2 - 2B^{-1}x) - 4\mathfrak{R}(2 - 2B^{-1}x - \frac{1}{2})$  (4.115)  
= 0.

Combining this, (4.112), and (4.113) with induction establishes item (ii). In addition, observe that (4.101), (4.103), (4.106), (4.107), and (4.108) show that for all  $n \in \mathbb{N}_0$  it holds that

$$\mathbb{S}_0(\boldsymbol{\ell}_n) = \mathbb{S}_0(\boldsymbol{\ell}_0) = \max\{2B^{-1}, 1\} \quad \text{and} \quad \mathbb{S}_1(\boldsymbol{\ell}_n) \le \max\{\mathbb{S}_1(\boldsymbol{\varrho}), \mathbb{S}_1(\boldsymbol{\ell}_0)\} = 2. \quad (4.116)$$

Furthermore, note that (4.101), (4.103), (4.106), (4.107), and (4.108) show that for all  $n \in \mathbb{N}_0$  it holds that

$$\mathcal{S}(\boldsymbol{f}_n) \le \max\{\mathcal{S}(\boldsymbol{g}), \mathcal{S}(\boldsymbol{f}_0), \|W\|_{\infty}, \|\mathfrak{W}\|_{\infty}\} = \max\{2, 2B^{-1}, 2, 2\} = \max\{2B^{-1}, 2\} \quad (4.117)$$

(cf. Definitions 2.12 and 2.13). This and (4.116) establish item (v). The proof of Lemma 4.18 is thus complete.  $\hfill \Box$ 

**Proposition 4.19.** Let  $n, N \in \mathbb{N}$ ,  $a, b \in [0, 2\pi]$ ,  $c \in \mathbb{R}$  satisfy  $b = a + \frac{4\pi}{N+1}$ , let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}$ ,  $x \in [2k-1, 2k+1)$  that  $\mathfrak{s}(x) = 1 - |x-2k|$ , let  $f \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in [0, 2\pi)$ ,  $y \in (-\infty, -2^n\pi) \cup [2^n\pi, \infty)$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  that

$$f(y) = 0 \qquad and \qquad f(x + 2k\pi) = f(x) = \begin{cases} c \mathfrak{s}\left(\frac{(x-a)(N+1)}{2\pi}\right) & : x \in [a,b] \\ 0 & : x \notin [a,b], \end{cases}$$
(4.118)

let  $a, \ell, c \in \mathbb{R}$  satisfy  $a = a - \frac{(N-1)\pi}{N+1}$ ,  $\ell = \frac{N-1}{N+1}$ , and  $c = \frac{(N+1)c}{2}$ , and let  $g: \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that

$$g(x) = \begin{cases} \mathcal{O}\mathfrak{R}\big(\mathfrak{s}(\pi^{-1}(x-a)+\mathscr{O})-\mathscr{O}\big) & : x \in [-2^n\pi+a, 2^n\pi+a] \\ 0 & : x \notin [-2^n\pi+a, 2^n\pi+a] \end{cases}$$
(4.119)

(cf. Definition 2.2). Then f = g.

Proof of Proposition 4.19. Observe that (4.119) and the fact that for all  $x \in [0, \infty)$  it holds that  $\mathfrak{s}(x) \leq x$  imply that for all  $x \in [a, a]$  it holds that

$$g(x) = \mathfrak{OR}\left(\mathfrak{s}(\pi^{-1}(x-a) + \mathscr{O}) - \mathscr{O}\right) = 0.$$
(4.120)

Furthermore, note that (4.119) and the fact for all  $x \in [0, 1]$  it holds that  $\mathfrak{s}(x) = x = \mathfrak{R}(x)$  demonstrates that for all  $x \in [a, a + \frac{2\pi}{N+1}]$  it holds that

$$g(x) = c \Re \left( \mathfrak{s} (\pi^{-1}(x-a) + \mathscr{O}) - \mathscr{O} \right) = c \Re \left( \mathfrak{s} (\pi^{-1}(x-a) + \frac{N-1}{N+1}) - \frac{N-1}{N+1} \right) = c \Re \left( \pi^{-1}(x-a) + \frac{N-1}{N+1} - \frac{N-1}{N+1} \right) = \left( \frac{2c}{N+1} \right) \Re \left( \frac{(x-a)(N+1)}{2\pi} \right) = c \Re \left( \frac{(x-a)(N+1)}{2\pi} \right) = c \mathfrak{s} \left( \frac{(x-a)(N+1)}{2\pi} \right).$$
(4.121)

Moreover, observe that (4.119) and the fact for all  $x \in [1, 2]$  it holds that  $\mathfrak{s}(x) = \mathfrak{s}(2 - x) = 2 - x$  and the fact that for all  $x \in [a + \frac{2\pi}{N+1}, b]$  it holds that  $\pi^{-1}(x - a) \in [\frac{2}{N+1}, \frac{4}{N+1}]$  and  $b \leq a + \frac{(N+3)\pi}{N+1} = a + 2\pi$  ensure that for all  $x \in [a + \frac{2\pi}{N+1}, b]$  it holds that

$$g(x) = c \Re \left( \mathfrak{s} \left( \pi^{-1} (x - a) + \mathscr{O} \right) - \mathscr{O} \right) \\ = c \Re \left( \mathfrak{s} \left( \pi^{-1} (x - a) + \frac{N - 1}{N + 1} \right) - \frac{N - 1}{N + 1} \right) \\ = c \Re \left( \mathfrak{s} \left( \pi^{-1} (x - a) - \frac{N - 1}{N + 1} - \frac{N - 1}{N + 1} \right) \right) \\ = c \Re \left( 2 - \pi^{-1} (x - a) \right) \\ = c \Re \left( \frac{4}{N + 1} - \pi^{-1} (x - a) \right) \\ = \left( \frac{2c}{N + 1} \right) \Re \left( 2 - \frac{(x - a)(N + 1)}{2\pi} \right) \\ = c \Re \left( 2 - \frac{(x - a)(N + 1)}{2\pi} \right) \\ = c \left( 2 - \frac{(x - a)(N + 1)}{2\pi} \right) \\ = c \mathfrak{s} \left( \frac{(x - a)(N + 1)}{2\pi} \right).$$
(4.122)

In addition, note that (4.119) and the fact that for all  $x \in \left[\frac{4}{N+1}, \frac{N+3}{N+1}\right]$  it holds that  $\mathfrak{s}(x+\mathscr{C}) \leq \mathfrak{s}\left(\frac{N+3}{N+1}\right) = \mathscr{C}$  show that for all  $x \in [b, 2\pi + \alpha] = \left[a + \frac{4\pi}{N+1}, a + \frac{(N+3)\pi}{N+1}\right]$  it holds that

$$g(x) = \mathfrak{c}\mathfrak{R}\big(\mathfrak{s}(\pi^{-1}(x-a) + \mathscr{C}) - \mathscr{C}\big) = 0.$$
(4.123)

Combining this, (4.118), (4.119), (4.120), (4.121), and (4.122) with the fact that  $\alpha \leq a$  implies that for all  $x \in [a, b], y \in [\alpha, a) \cup (b, 2\pi + \alpha]$  it holds that

$$g(x) = c \mathfrak{s}\left(\frac{(x-a)(N+1)}{2\pi}\right) = f(x)$$
 and  $g(y) = 0 = f(y).$  (4.124)

Furthermore, observe that (4.119) and the fact that for all  $x \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  it holds that  $\mathfrak{s}(x+2k) = \mathfrak{s}(x)$  show that for all  $x \in [\mathfrak{a}, 2\pi + \mathfrak{a}]$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  it holds that

$$g(x+2k\pi) = c\Re(\mathfrak{s}(\pi^{-1}(x-a)+2k)-\ell) = c\Re(\mathfrak{s}(\pi^{-1}(x-a))-\ell) = g(x).$$
(4.125)

This and (4.118) demonstrate that for all  $x \in [a, a+2\pi)$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  with  $x+2k\pi \in [\max\{a, 0\}-2^n\pi, \min\{a, 0\}+2^n\pi)$  it holds that

$$g(x+2k\pi) = g(x)$$
 and  $f(x+2k\pi) = f(x)$ . (4.126)

This and (4.124) show that for all  $x \in [\max\{a, 0\} - 2^n \pi, \min\{a, 0\} + 2^n \pi)$  it holds that

$$g(x) = f(x).$$
 (4.127)

Moreover, note that (4.118), (4.119), and the fact that  $2\pi \ge b$  and  $\alpha \le a$  imply that for all  $x \in \mathbb{R}$  with  $-2^n \pi \le x \le \alpha - 2^n \pi$  it holds that

$$f(x) = f(x + 2^n \pi) = 0 = g(x).$$
(4.128)

In addition, observe that (4.118), (4.120), (4.125), and the fact that  $\max\{\alpha, 0\} \leq a$  imply that for all  $x \in \mathbb{R}$  with  $\alpha - 2^n \pi \leq x \leq -2^n \pi$  it holds that

$$g(x) = g(x + 2^n \pi) = 0 = f(x).$$
(4.129)

Furthermore, note that (4.118), (4.119), and the fact that  $a \ge b - 2\pi$  and  $0 \le a$  imply that for all  $x \in \mathbb{R}$  with  $a + 2^n \pi \le x \le 2^n \pi$  it holds that

$$f(x) = f(x - 2^n \pi) = 0 = g(x).$$
(4.130)

Moreover, observe that (4.118), (4.120), (4.124), and the fact that  $2\pi \ge b$  imply that for all  $x \in \mathbb{R}$  with  $2^n \pi \le x \le a + 2^n \pi$  it holds that

$$g(x) = g(x - 2(2^{n-1} - 1)\pi) = 0 = f(x).$$
(4.131)

Combining this, (4.128), (4.129), and (4.130) with (4.118) and (4.119) ensures that for all  $x \in (-\infty, \max\{\alpha, 0\} - 2^n \pi], y \in [\min\{\alpha, 0\} + 2^n \pi, \infty)$  it holds that

$$f(x) = g(x) = 0 = f(y) = g(y).$$
 (4.132)

This and (4.127) establish f = g. The proof of Proposition 4.19 is thus complete.

**Lemma 4.20.** Let  $n, N \in \mathbb{N}$ ,  $C \in [1, \infty)$ ,  $a, b \in [0, 2\pi]$ ,  $c \in [-C, C]$  satisfy  $b = a + \frac{4\pi}{N+1}$  and  $n \geq 2$ , let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}$ ,  $x \in [2k-1, 2k+1)$  that  $\mathfrak{s}(x) = 1 - |x-2k|$ , and let  $f \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in [0, 2\pi)$ ,  $y \in (-\infty, -2^n\pi) \cup [2^n\pi, \infty)$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  that

$$f(y) = 0 \qquad and \qquad f(x + 2k\pi) = f(x) = \begin{cases} c \mathfrak{s}\left(\frac{(x-a)(N+1)}{2\pi}\right) & : x \in [a,b] \\ 0 & : x \notin [a,b]. \end{cases}$$
(4.133)

Then there exists  $\not \in \mathbf{N}$  such that

- (i) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(\mathcal{I}))(x) = \left(\frac{2}{C(N+1)}\right)f(x)$ ,
- (ii) it holds that  $\mathcal{L}(\mathbf{p}) = n + 5$ ,

(iii) it holds for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{J})]$  that

$$\mathbb{D}_{k}(\mathscr{I}) = \begin{cases} 1 & : k \in \{0, n+4, n+5\} \\ 2 & : k \in \{1, 2, 3, n+3\} \\ 4 & : k \in \mathbb{N} \cap (3, n+3), \end{cases}$$
(4.134)

and

(iv) it holds that  $\mathcal{S}(\not e) \leq 2$ 

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Lemma 4.20. Throughout this proof let  $\alpha \in [-\pi, a], \ \ell \in [0, 1], \ c \in \mathbb{R}, \ n \in \mathbb{N}_0$  satisfy

$$a = a - \frac{(N-1)\pi}{N+1}, \quad b = \frac{N-1}{N+1}, \quad c = \frac{c}{C}, \quad \text{and} \quad n = \lceil \log_2((N+1)C) \rceil - 1, \quad (4.135) \rceil$$

and let  $g_1 \in ((\mathbb{R}^{2 \times 1} \times \mathbb{R}^2) \times (\mathbb{R}^{1 \times 2} \times \mathbb{R}^1)) \subseteq \mathbb{N}$  and  $g_3 \in ((\mathbb{R}^{1 \times 1} \times \mathbb{R}^1) \times (\mathbb{R}^{1 \times 1} \times \mathbb{R}^1)) \subseteq \mathbb{N}$  satisfy

$$g_1 = \left( \left( \begin{pmatrix} 1\\-1 \end{pmatrix}, \begin{pmatrix} -\alpha 2^{-1}\\ \alpha 2^{-1} \end{pmatrix} \right), \left( \begin{pmatrix} 1\\-1 \end{pmatrix}, -\alpha 2^{-1} \end{pmatrix} \right) \quad \text{and} \quad g_3 = \left( (1, -\mathscr{C}), (\varepsilon, 0) \right) \quad (4.136)$$

(cf. Definitions 2.1 and 4.8). Note that Lemma 4.18 (applied with  $B \curvearrowleft 2^n \pi$ ,  $n \backsim n-1$  in the notation of Lemma 4.18) implies that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in [0, 2^n \pi]$  that  $(\mathcal{R}(\mathcal{G}_2))(-x) = (\mathcal{R}(\mathcal{G}_2))(x) = \mathfrak{s}(2^n (2^n \pi)^{-1} x) = \mathfrak{s}(\pi^{-1} x),$
- (II) it holds for all  $x \in (2^n \pi, \infty)$  that  $(\mathcal{R}(g_2))(-x) = (\mathcal{R}(g_2))(x) = 0$ ,
- (III) it holds that  $\mathcal{L}(g_2) = (n-1) + 2$ ,
- (IV) it holds that  $\mathcal{D}(g_2) = (1, 4, 4, \dots, 4, 1) \in \mathbb{N}^{n+2}$ , and
- (V) it holds that  $\mathcal{S}(g_2) \leq 2$ ,  $\mathbb{S}_0(g_2) = 1$ , and  $\mathbb{S}_1(g_2) \leq 2$
- (cf. Definitions 2.3 and 2.13). Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{\ell} = \boldsymbol{g}_3 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1. \tag{4.137}$$

Observe that (4.136) ensures that for all  $x \in \mathbb{R}$  it holds that  $\mathcal{S}(g_1) \leq \pi 2^{-1} \leq 2$ ,  $\mathcal{D}(g_1) = (1, 2, 1)$ , and

$$(\mathcal{R}(g_1))(x) = \Re(x - a2^{-1}) - \Re(-(x - a2^{-1})) - a2^{-1} = x - a.$$
(4.138)

Furthermore, note that (4.136) ensures that for all  $x \in \mathbb{R}$  it holds that

$$\mathcal{S}(g_3) \le 1, \qquad \mathcal{D}(g_3) = (1, 1, 1), \qquad \text{and} \qquad (\mathcal{R}(g_3))(x) = c \mathfrak{R}(x - \ell).$$
 (4.139)

Combining this and (4.138) with Proposition 2.10 and Proposition 2.7 implies that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathscr{f})(x) = ([\mathcal{R}(\mathscr{g}_3)] \circ [\mathcal{R}(\mathscr{g}_2)] \circ [\mathcal{R}(\mathscr{g}_1)])(x) = \mathscr{C}\mathfrak{R}((\mathcal{R}(\mathscr{g}_2))(x-\mathscr{a}) - \mathscr{C}).$$
(4.140)

This and item (I) show that for all  $x \in [-2^n \pi + a, 2^n \pi + a]$  it holds that

$$(\mathcal{R}(\boldsymbol{f}))(x) = c \Re ((\mathcal{R}(\boldsymbol{g}_2))(x-a) - \boldsymbol{\ell}) = c \Re (\mathfrak{s}(\pi^{-1}x - \pi^{-1}a) - \boldsymbol{\ell}) = c \Re (\mathfrak{s}(\pi^{-1}x - (\pi^{-1}a - \boldsymbol{\ell})) - \boldsymbol{\ell}) = c \Re (\mathfrak{s}(\pi^{-1}(x-a) + \boldsymbol{\ell}) - \boldsymbol{\ell}) = (\frac{2}{(N+1)C}) (\frac{(N+1)c}{2}) \Re (\mathfrak{s}(\pi^{-1}(x-a) + \boldsymbol{\ell}) - \boldsymbol{\ell}).$$
(4.141)

Moreover, observe that item (II) and (4.140) demonstrate that for all  $x \in \mathbb{R} \setminus [-2^n \pi + a, 2^n \pi + a]$  it holds that

$$(\mathcal{R}(f))(x) = c \Re ((\mathcal{R}(g_2))(x-a) - b) = c \Re (-b) = 0.$$
(4.142)

Combining this, (4.133), and (4.142) with Proposition 4.19 (applied with  $n \curvearrowleft n, N \curvearrowleft N$ ,  $a \curvearrowleft a, b \curvearrowleft b, c \backsim c, f \backsim f, a \curvearrowleft a, b \backsim b, c \curvearrowleft \frac{(N+1)c}{2}, g \backsim \left(\frac{C(N+1)}{2}\right) \mathcal{R}(f)$  in the notation of Proposition 4.19) shows that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathbf{f}))(x) = \left(\frac{2}{C(N+1)}\right) \left(\frac{C(N+1)}{2}\right) (\mathcal{R}(\mathbf{f}))(x) = \left(\frac{2}{C(N+1)}\right) f(x).$$
(4.143)

Note that Proposition 2.10, Proposition 2.7, (4.136), and item (III), show that

$$\mathcal{L}(\mathbf{f}) = \mathcal{L}(\mathbf{g}_3) + \mathcal{L}(\mathbf{g}_2) + \mathcal{L}(\mathbf{g}_1) + 2\mathcal{L}(\mathbb{I}_1) - 4 = 2 + (n+1) + 2 = n+5.$$
(4.144)

This, (4.136), (4.137), (4.138), (4.139), item (IV), Lemma 2.11, and Proposition 2.7 demonstrate that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{I})]$  it holds that

$$\mathbb{D}_{k}(\mathscr{I}) = \begin{cases} 1 & : k \in \{0, n+4, n+5\} \\ 2 & : k \in \{1, 2, n+3\} \\ 4 & : k \in \mathbb{N} \cap (2, n+3). \end{cases}$$
(4.145)

In addition, observe that (4.135), (4.136), (4.137), (4.139), item (V), and Proposition 2.18 show that

$$\mathcal{S}(\boldsymbol{\ell}) \leq \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \leq \max\{1, 2, 2\} = 2.$$
(4.146)

Combining this, (4.143), (4.144), and (4.145), establishes items (i), (ii), and (iii). The proof of Lemma 4.20 is thus complete.

**Lemma 4.21.** Let  $n, N \in \mathbb{N} \cap (1, \infty)$ ,  $C \in [1, \infty)$ ,  $\lambda \in [-2, 2]$ , let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}$ ,  $x \in [2k-1, 2k+1)$  that  $\mathfrak{s}(x) = 1 - |x-2k|$ , for every  $j \in \{1, 2, \dots, N\}$  let  $a_j, b_j \in [0, 2\pi]$ ,  $c_j \in [-C, C]$  satisfy  $b_j = a_j + \frac{4\pi}{N+1}$ , and for every  $j \in \{1, 2, \dots, N\}$  let  $f_j \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in [0, 2\pi)$ ,  $y \in (-\infty, -2^n\pi) \cup [2^n\pi, \infty)$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$ , that

$$f_j(y) = 0 \quad and \quad f_j(x+2k\pi) = f_j(x) = \begin{cases} c_j \mathfrak{s}\left(\frac{(x-a_j)(N+1)}{2\pi}\right) & : x \in [a_j, b_j] \\ 0 & : x \notin [a_j, b_j]. \end{cases}$$
(4.147)

Then there exists  $\not \in \mathbf{N}$  such that

- (i) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(\mathcal{I}))(x) = \lambda + \sum_{j=1}^{N} f_j(x)$ ,
- (ii) it holds that  $\mathcal{L}(\not l) \leq n + \log_2(C) + 9$ ,
- (iii) it holds that  $\mathbb{D}_0(\mathcal{F}) = \mathbb{D}_{\mathcal{L}(\mathcal{F})}(\mathcal{F}) = 1$ ,  $\mathbb{D}_1(\mathcal{F}) = 2N$ , and  $\mathbb{D}_{\mathcal{H}(\mathcal{F})}(\mathcal{F}) = 2$ ,
- (iv) it holds that  $\mathcal{P}(\not{e}) \leq (24 + 18n + 5\log_2(C))N^2$ , and
- (v) it holds that  $\mathcal{S}(\not e) \leq 2$
- (cf. Definitions 2.1, 2.3, and 2.13).

Proof of Lemma 4.21. Throughout this proof let  $n, B \in \mathbb{N}, \beta \in (0, 2]$  satisfy

$$n = \min\{\mathbb{N} \cap [\log_2(C), \infty)\}, \quad B = \left\lceil \left(\frac{N+1}{2}\right)^{\frac{1}{n}} \right\rceil, \quad \text{and} \quad \beta = B^{-1} \left(\frac{C(N+1)}{2}\right)^{\frac{1}{n}}, \quad (4.148)$$

and let  $g_1, g_2, g_4 \in \mathbf{N}$  satisfy  $g_4 = ((1), \lambda) \in (\mathbb{R}^{1 \times 1} \times \mathbb{R}^1)$ ,

$$g_1 = \left( \begin{pmatrix} 1\\1\\\vdots\\1 \end{pmatrix}, 0 \right) \in (\mathbb{R}^{N \times 1} \times \mathbb{R}^N), \text{ and } g_2 = \left( \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}, 0 \right) \in (\mathbb{R}^{1 \times N} \times \mathbb{R}^1) \quad (4.149)$$

(cf. Definitions 2.1 and 4.8). Note that Lemma 4.20 (applied with  $n \curvearrowleft n, N \curvearrowleft N, C \curvearrowleft C, a \curvearrowleft a_j, b \curvearrowleft b_j, c \curvearrowleft c_j, f \curvearrowleft f_j$  for  $j \in \{1, 2, \ldots, N\}$  in the notation of Lemma 4.20) implies that there exist  $f_1, f_2, \ldots, f_N \in \mathbb{N}$  such that

- (I) it holds for all  $j \in \{1, 2, ..., N\}$ ,  $x \in \mathbb{R}$  that  $(\mathcal{R}(\mathcal{F}_j))(x) = \left(\frac{2}{C(N+1)}\right) f_j(x)$ ,
- (II) it holds for all  $j \in \{1, 2, \dots, N\}$  that  $\mathcal{L}(\mathcal{L}_j) = n + 5$ ,
- (III) it holds for all  $j \in \{1, 2, ..., N\}$  that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{F}_j)]$  that

$$\mathbb{D}_{k}(\mathscr{J}_{j}) = \begin{cases} 1 & : k \in \{0, n+4, n+5\} \\ 2 & : k \in \{1, 2, n+3\} \\ 4 & : k \in \mathbb{N} \cap (2, n+3), \end{cases}$$
(4.150)

and

(IV) it holds for all  $j \in \{1, 2, ..., N\}$  that  $\mathcal{S}(\mathbf{p}_j) \leq 2$ 

(cf. Definitions 2.3 and 2.13). Furthermore, observe that Lemma 4.5 (applied with  $\beta \curvearrowleft \beta$ ,  $B \curvearrowleft B$ ,  $n \curvearrowleft n$  in the notation of Lemma 4.5) demonstrates that there exists  $g_3 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_3))(x) = \frac{C(N+1)x}{2}$ ,
- (B) it holds that  $\mathcal{D}(g_3) = (1, 2B, 2B, \dots, 2B, 1) \in \mathbb{N}^{n+2}$ ,

- (C) it holds that  $\mathbb{S}_0(\mathfrak{g}_3) \leq 1$ ,  $\mathbb{S}_1(\mathfrak{g}_3) \leq 2$ , and  $\mathcal{S}(\mathfrak{g}_3) \leq 2$ , and
- (D) it holds that  $\mathcal{P}(g_3) = (4n 4)B^2 + (2n + 4)B + 1$ .

Next let  $\mathscr{R} \in \mathbf{N}$  satisfy

$$\hbar = g_4 \bullet \mathbb{I}_1 \bullet g_3 \bullet g_2 \bullet \mathbb{I}_N \bullet (\mathbf{P}_N(\not_1, \not_2, \dots, \not_N)) \bullet \mathbb{I}_N \bullet g_1$$
(4.151)

(cf. Definitions 2.4, 2.6, and 2.8). Note that (4.149), (4.151), Proposition 2.10, and Proposition 2.5 imply that for all  $x \in \mathbb{R}$  it holds that

$$\begin{aligned} (\mathcal{R}(\mathscr{A}))(x) \\ &= ([\mathcal{R}(\mathscr{g}_4)] \circ [\mathcal{R}(\mathscr{g}_3)] \circ [\mathcal{R}(\mathscr{g}_2)] \circ [\mathcal{R}(\mathbf{P}_N(\mathscr{f}_1, \mathscr{f}_2, \dots, \mathscr{f}_N))])((\mathcal{R}(\mathscr{g}_1))(x)) \\ &= ([\mathcal{R}(\mathscr{g}_4)] \circ [\mathcal{R}(\mathscr{g}_3)] \circ [\mathcal{R}(\mathscr{g}_2)] \circ [\mathcal{R}(\mathbf{P}_N(\mathscr{f}_1, \mathscr{f}_2, \dots, \mathscr{f}_N))])(x, x, \dots, x) \\ &= ([\mathcal{R}(\mathscr{g}_4)] \circ [\mathcal{R}(\mathscr{g}_3)] \circ [\mathcal{R}(\mathscr{g}_2)]) \big( (\mathcal{R}(\mathbf{P}_N(\mathscr{f}_1, \mathscr{f}_2, \dots, \mathscr{f}_N)))(x, x, \dots, x) \big) \\ &= ([\mathcal{R}(\mathscr{g}_4)] \circ [\mathcal{R}(\mathscr{g}_3)] \circ [\mathcal{R}(\mathscr{g}_2)]) \big( (\mathcal{R}(\mathscr{f}_1))(x), (\mathcal{R}(\mathscr{f}_2))(x), \dots, (\mathcal{R}(\mathscr{f}_N))(x) \big) \\ &= ([\mathcal{R}(\mathscr{g}_4)] \circ [\mathcal{R}(\mathscr{g}_3)]) \Big( \sum_{j=1}^N (\mathcal{R}(\mathscr{f}_j))(x) \Big) \\ &= (\mathcal{R}(\mathscr{g}_4)) \Big( \Big( \frac{C(N+1)}{2} \Big) \sum_{j=1}^N \Big( \frac{2}{C(N+1)} \Big) f_j(x) \Big) = \lambda + \sum_{j=1}^N f_j(x). \end{aligned}$$

Observe that (4.149), (4.151), item (II), item (B), Proposition 2.10, and Proposition 2.5 ensure that

$$\mathcal{L}(\hbar) = \mathcal{L}(g_4) + \mathcal{L}(g_3) + \mathcal{L}(g_2) + \mathcal{L}(\mathbf{P}_N(\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N)) + \mathcal{L}(g_1) + 3\mathcal{L}(\mathbb{I}_1) - 7$$
  
= 1 +  $\mathcal{L}(g_3) + 1 + \mathcal{L}(\mathbf{f}_1) + 1 + 6 - 7$   
= (n + 5) + (n + 1) + 2 \le n + log\_2(C) + 9. (4.153)

Note that item (II), item (III), and Proposition 2.5 imply that for all  $k \in \mathbb{N}_0 \cap [0, n+5]$  it holds that

$$\mathbb{D}_{k}(\mathbf{P}_{N}(\mathbf{\ell}_{1},\mathbf{\ell}_{2},\ldots,\mathbf{\ell}_{N})) = \begin{cases} N & : k \in \{0, n+4, n+5\} \\ 2N & : k \in \{1, 2, n+3\} \\ 4N & : k \in \mathbb{N} \cap (2, n+3). \end{cases}$$
(4.154)

Combining this, (4.151), (4.153), item (B), Lemma 2.11, and Proposition 2.7 with the fact that  $\mathcal{D}(g_1) = (1, N)$ ,  $\mathcal{D}(g_2) = (N, 1)$ , and  $\mathcal{D}(g_4) = (1, 1)$  ensures that for all  $k \in \mathbb{N}_0 \cap [0, n + n + 8]$  it holds that

$$\mathbb{D}_{k}(\mathscr{R}) = \begin{cases} 1 & : k \in \{0, n+n+8\} \\ 2N & : k \in \{1, 2, 3, n+4, n+6\} \\ 4N & : k \in \mathbb{N} \cap (3, n+4) \\ N & : k \in \{n+5\} \\ 2B & : k \in \mathbb{N} \cap (n+6, n+n+7) \\ 2 & : k = n+n+7. \end{cases}$$
(4.155)

This and the fact that  $B \leq \left\lceil \frac{N+1}{2} \right\rceil \leq N$  demonstrates that

$$\begin{aligned} \mathcal{P}(\mathscr{R}) &= \sum_{k=1}^{n+n+8} \mathbb{D}_k(\mathscr{f})(\mathbb{D}_{k-1}(\mathscr{f})+1) \\ &= 2N(1+1) + 2(2N(2N+1)) + 4N(2N+1) + \left[\sum_{k=5}^{n+3} 4N(4N+1)\right] \\ &+ 2N(4N+1) + N(2N+1) + 2N(N+1) + 2B(2N+1) \\ &+ \left[\sum_{k=3}^{n+1} 2B(2B+1)\right] + 2(2B+1) + 1(2+1) \end{aligned}$$
(4.156)  
$$&= 28N^2 + 17N + 2BN + 4B + 5 + (n-1)(16N^2 + 4N) + (n-1)(4B^2 + 2B) \\ &\leq 30N^2 + 21N + 5 + (n-1)(16N^2 + 4N) + (n-1)(4N^2 + 2N) \\ &\leq 14N^2 + 17N + 5 + n(16N^2 + 4N) + \log_2(C)(4N^2 + 2N) \\ &\leq (24 + 18n + 5\log_2(C))N^2. \end{aligned}$$

Moreover, observe that (4.149), (4.151), item (IV), item (C), Lemma 2.16, Lemma 2.14, and Proposition 2.18 show that

$$\begin{aligned} \mathcal{S}(\hbar) \\ &\leq \max\{\mathcal{S}(\boldsymbol{g}_{4}), \mathcal{S}(\boldsymbol{g}_{3} \bullet \boldsymbol{g}_{2}), \mathcal{S}(\mathbf{P}_{N}(\boldsymbol{f}_{1}, \boldsymbol{f}_{2}, \dots, \boldsymbol{f}_{N})), \mathcal{S}(\boldsymbol{g}_{1})\} \\ &= \max\{\mathcal{S}(\boldsymbol{g}_{4}), \mathcal{S}(\boldsymbol{g}_{3}), \mathcal{S}(\boldsymbol{g}_{2}), \mathbb{S}_{0}(\boldsymbol{g}_{3})(\mathbb{S}_{1}(\boldsymbol{g}_{2})+1), \mathcal{S}(\boldsymbol{f}_{1}), \mathcal{S}(\boldsymbol{f}_{2}), \dots, \mathcal{S}(\boldsymbol{f}_{N}), \mathcal{S}(\boldsymbol{g}_{1})\} \\ &\leq \max\{\lambda, 2, 1, 2, 2, 2, \dots, 2, 1\} = 2. \end{aligned} \tag{4.157}$$

Combining this, (4.152), (4.153), (4.155), and (4.156) establishes items (i), (ii), (iii), (iv), and (v). The proof of Lemma 4.21 is thus complete.

**Lemma 4.22.** Let  $\varepsilon \in (0,1)$ ,  $n, N \in \mathbb{N}$  satisfy  $\varepsilon(N+1) \geq 2\pi$ , let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}, x \in [2k-1, 2k+1)$  that  $\mathfrak{s}(x) = 1 - |x-2k|$ , for every  $j \in \mathbb{N}_0 \cap [0, N+1]$  let  $c_j = \frac{2j\pi}{N+1}$ , let  $g \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x, y \in \mathbb{R}, k \in \mathbb{Z}$  that

$$|g(0)| \le 2,$$
  $g(x+2k\pi) = g(x),$  and  $|g(x) - g(y)| \le |x-y|,$  (4.158)

for every  $j \in \{1, 2, ..., N\}$  let  $f_j \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in [0, 2\pi)$ ,  $y \in (-\infty, -2^n \pi) \cup [2^n \pi, \infty)$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  that  $f_j(y) = 0$  and

$$f_j(x+2k\pi) = f_j(x) = \begin{cases} (g(c_j) - g(0))\mathfrak{s}\left(\frac{(x-c_{j-1})(N+1)}{2\pi}\right) & : x \in [c_{j-1}, c_{j+1}] \\ 0 & : x \notin [c_{j-1}, c_{j+1}], \end{cases}$$
(4.159)

and let  $F \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that

$$F(x) = g(0) + \sum_{j=1}^{N} f_j(x).$$
(4.160)

Then

- (i) it holds that  $\sup_{x \in [-2^n \pi, 2^n \pi]} |g(x) F(x)| \le \varepsilon$  and
- (ii) it holds for all  $x, y \in \mathbb{R}$  that  $|F(x) F(y)| \le |x y|$ .

Proof of Lemma 4.22. Note that (4.159) and (4.160) imply that for all  $x \in (2^n \pi, \infty)$  it holds that

$$F(-x) = g(0) = F(x).$$
(4.161)

Furthermore, observe that (4.159) and the fact that  $\mathfrak{s}(0) = 0 = \mathfrak{s}(2)$  show that for all  $j \in \{1, 2, \ldots, N\}$  it holds that

$$f_j(c_{j-1}) = (g(c_j) - g(0))\mathfrak{s}(0) = 0 = (g(c_j) - g(0))\mathfrak{s}(2) = f_j(c_{j+1}).$$
(4.162)

This, (4.159), (4.160), and the fact that for all  $x \in [0, 1]$  it holds that  $\mathfrak{s}(x) = x$  imply that for all  $x \in [c_0, c_1]$  it holds that

$$F(x) = g(0) + \sum_{k=1}^{N} f_k(x) = g(0) + f_1(x) = g(0) + (g(c_1) - g(0))\mathfrak{s}\left(\frac{(x - c_0)(N + 1)}{2\pi}\right)$$

$$= g(0) + (g(c_1) - g(0))\left(\frac{(N + 1)x}{2\pi}\right).$$
(4.163)

Moreover, note that (4.159), (4.160), (4.162), and the fact that for all  $x \in [1, 2]$  it holds that  $\mathfrak{s}(x) = 2 - x$  imply that for all  $x \in [c_N, c_{N+1}]$  it holds that

$$F(x) = g(0) + \sum_{k=1}^{N} f_k(x) = g(0) + f_N(x)$$
  
=  $g(0) + (g(c_N) - g(0)) \mathfrak{s} \left( \frac{(x - c_{N-1})(N+1)}{2\pi} \right)$   
=  $g(0) + (g(c_N) - g(0)) \left( 2 - \frac{(x - c_{N-1})(N+1)}{2\pi} \right).$  (4.164)

In addition, observe that (4.159), (4.160), (4.162), and the fact that for all  $x \in [0, 1]$  it holds that  $\mathfrak{s}(x) = x$  and  $\mathfrak{s}(x+1) = 2 - (x+1)$  ensure that for all  $j \in \mathbb{N} \cap (1, N]$ ,  $x \in [c_{j-1}, c_j]$  it holds that

$$F(x) = g(0) + \sum_{k=1}^{N} f_k(x)$$
  

$$= g(0) + f_{j-1}(x) + f_j(x)$$
  

$$= g(0) + (g(c_{j-1}) - g(0)) \mathfrak{s} \left( \frac{(x - c_{j-2})(N+1)}{2\pi} \right)$$
  

$$+ (g(c_j) - g(0)) \mathfrak{s} \left( \frac{(x - c_{j-1})(N+1)}{2\pi} \right)$$
  

$$= g(0) + (g(c_{j-1}) - g(0)) \left( 2 - \frac{(x - c_{j-2})(N+1)}{2\pi} \right)$$
  

$$+ (g(c_j) - g(0)) \left( \frac{(x - c_{j-1})(N+1)}{2\pi} \right).$$
  
(4.165)

Combining this and (4.163) with (4.164) implies that for all  $j \in \{1, 2, ..., N+1\}$  it holds that

$$F|_{[c_{j-1},c_j]} \in \mathfrak{L}([c_{j-1},c_j])$$
 (4.166)

(cf. Definition 3.5). Furthermore, note that (4.159), (4.160), (4.162), and the fact that  $\mathfrak{s}(1) = 1$  ensure that for all  $j \in \{1, 2, \ldots, N\}$  it holds that

$$F(c_j) = g(0) + \sum_{k=1}^{N} f_k(c_j) = g(0) + f_j(c_j) = g(0) + (g(c_j) - g(0))\mathfrak{s}(1) = g(c_j).$$
(4.167)

This, (4.163), (4.164), (4.165), (4.166), and the fact that F(0) = g(0) and  $F(2\pi) = g(0) = g(2\pi)$  demonstrate that for all  $j \in \{1, 2, ..., N+1\}, x \in [c_{j-1}, c_j]$  it holds that

$$F(x) = \left(\frac{2j\pi - (N+1)x}{2\pi}\right)F(c_j) + \left(\frac{(N+1)x - 2(j-1)\pi}{2\pi}\right)F(c_{j-1}).$$
(4.168)

Combining this and (4.167) with the assumption that for all  $x, y \in \mathbb{R}$  it holds that  $|g(x)-g(y)| \leq |x-y|$  and  $\varepsilon(N+1) \geq 2\pi$  demonstrates that for all  $j \in \{1, 2, \ldots, N+1\}, x \in [c_{j-1}, c_j]$  it holds that

$$|g(x) - F(x)| = \left| g(x) - \left(\frac{2j\pi - (N+1)x}{2\pi}\right) F(c_j) - \left(\frac{(N+1)x - 2(j-1)\pi}{2\pi}\right) F(c_{j-1}) \right|$$
  

$$= \left| \left(\frac{2j\pi - (N+1)x}{2\pi}\right) (g(x) - F(c_j)) + \left(\frac{(N+1)x - 2(j-1)\pi}{2\pi}\right) (g(x) - F(c_{j-1})) \right|$$
  

$$= \left| \left(\frac{2j\pi - (N+1)x}{2\pi}\right) (g(x) - g(c_j)) + \left(\frac{(N+1)x - 2(j-1)\pi}{2\pi}\right) (g(x) - g(c_{j-1})) \right|$$
  

$$\leq \left| \left(\frac{2j\pi - (N+1)x}{2\pi}\right) (x - c_j) \right| + \left| \left(\frac{(N+1)x - 2(j-1)\pi}{2\pi}\right) (x - c_{j-1}) \right|$$
  

$$\leq \left| \left(\frac{2j\pi - (N+1)x}{2\pi}\right) \frac{2\pi}{N+1} \right| + \left| \left(\frac{(N+1)x - 2(j-1)\pi}{2\pi}\right) \frac{2\pi}{N+1} \right| = \frac{2\pi}{N+1} \leq \varepsilon.$$

Moreover, observe that that (4.166), (4.167), and the assumption that for all  $j \in \{1, 2, ..., N\}$ ,  $x, y \in \mathbb{R}$  it holds that  $|g(x) - g(y)| \leq |x - y|$  ensure that for all  $x, y \in [0, 2\pi] = [c_0, c_{N+1}]$  it holds that

$$|F(x) - F(y)| \le |x - y| \max_{j \in \{1, 2, \dots, N+1\}} \frac{g(c_j) - g(c_{j-1})}{c_j - c_{j-1}} \le |x - y|.$$
(4.170)

Combining this and (4.161) with (4.169) and the fact that for all  $x \in [0, 2\pi]$ ,  $k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  it holds that  $g(x + 2k\pi) = g(x)$  and  $F(x + 2k\pi) = F(x)$  establishes items (i) and (ii). The proof of Lemma 4.22 is thus complete.

**Lemma 4.23.** Let  $\varepsilon \in (0,1)$ ,  $N \in \mathbb{N}$ ,  $C \in [1,\infty)$  satisfy  $\varepsilon(N+1) \ge 2\pi$  and let  $g: \mathbb{R} \to \mathbb{R}$ satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \in [-2,2]$ ,  $g(x+2k\pi) = g(x) \in [-C,C]$ , and  $|g(x) - g(y)| \le |x - y|$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{L}(\mathcal{L}) = 6$ ,
- (*ii*) it holds it holds that  $\mathcal{D}(\not l) = (1, 2, 2N + 3, 2, 2, 2, 1)$ ,
- (iii) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(\mathcal{I}))(x) (\mathcal{R}(\mathcal{I}))(y)| \le |x y|,$

- (iv) it holds that  $\sup_{x \in [-2\pi, 2\pi]} |g(x) (\mathcal{R}(\mathscr{I}))(x)| \le \varepsilon$ ,
- (v) it holds for all  $x \in [2\pi, \infty)$  that  $(\mathcal{R}(\mathbf{f}))(-x) = (\mathcal{R}(\mathbf{f}))(x) = g(0)$ ,
- (vi) it holds that  $\mathcal{S}(\mathbf{f}) \leq 2$ , and
- (vii) it holds that  $\mathcal{P}(\not) \leq 4N^2$
- (cf. Definitions 2.1, 2.3, and 2.13).

Proof of Lemma 4.23. Throughout this proof let  $M \in \mathbb{N}$ ,  $d \in (0, \varepsilon]$  satisfy M = 2N + 2 and  $d = \frac{4\pi}{M}$ , let  $\xi_0, \xi_1, \ldots, \xi_M \in [-2\pi, 2\pi]$ ,  $\alpha_0, \alpha_1, \ldots, \alpha_M \in [-2, 2]$  satisfy for all  $k \in \mathbb{N}_0 \cap [0, M]$  that

$$\xi_k = kd - 2\pi$$
 and  $\alpha_k = \frac{g(\xi_{\min\{k+1,M\}}) - g(\xi_k)}{d} - \frac{g(\xi_k) - g(\xi_{\max\{k-1,0\}})}{d},$  (4.171)

and let  $g_1 \in \left( (\mathbb{R}^{(M+1)\times 1} \times \mathbb{R}^{M+1}) \times (\mathbb{R}^{1\times (M+1)} \times \mathbb{R}^1) \right) \subseteq \mathbf{N}$  satisfy

$$g_{1} = \left( \left( \begin{pmatrix} 4^{-1} \\ 4^{-1} \\ \vdots \\ 4^{-1} \end{pmatrix}, \begin{pmatrix} 4^{-1}\xi_{0} \\ 4^{-1}\xi_{1} \\ \vdots \\ 4^{-1}\xi_{M} \end{pmatrix} \right), \left( (\alpha_{0} \quad \alpha_{1} \quad \cdots \quad \alpha_{M}), 4^{-1}g(0) \right) \right)$$
(4.172)

(cf. Definition 2.1). Note that Corollary 4.6 (applied with  $\beta \curvearrowleft 4$ ,  $L \curvearrowleft 2$  in the notation of Corollary 4.6) ensures that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_2))(x) = 4x$ ,
- (II) it holds that  $\mathcal{D}(g_2) = (1, 2, 2, 1) \in \mathbb{N}^4$ , and
- (III) it holds that  $\mathcal{S}(g_2) \leq 2$

(cf. Definitions 2.3 and 2.13). Next let  $\not \in \mathbb{N}$  satisfy

$$\boldsymbol{f} = \boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1 \bullet \mathbb{I}_1 \tag{4.173}$$

(cf. Definitions 2.6 and 2.8). Observe that Proposition 2.10, Proposition 2.7, and (4.173) demonstrate that

$$\mathcal{L}(\mathbf{f}) = \mathcal{L}(\mathbf{g}_1) + \mathcal{L}(\mathbf{g}_2) + 2\mathcal{L}(\mathbb{I}_1) - 3 = 2 + 3 + 4 - 3 = 6.$$
(4.174)

Note that (4.172) shows that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\mathcal{Q}_1))(x) = \frac{g(0)}{4} + \sum_{k=0}^{M} \alpha_k \Re\left(\frac{x}{4} - \frac{\xi_k}{4}\right) = 4^{-1} \left[g(0) + \sum_{k=0}^{M} \alpha_k \Re(x - \xi_k)\right]$$
(4.175)

(cf. Definition 2.2). This and (4.171) demonstrate that for all  $x \in (-\infty, \xi_0]$  it holds that

$$(\mathcal{R}(g_1))(x) = \frac{g(0)}{4} = \frac{g(\xi_0)}{4} \quad \text{and} \quad (\mathcal{R}(g_1))(\xi_1) = \frac{g(0) + \alpha_0 \Re(\xi_1 - \xi_0)}{4} = \frac{g(\xi_1)}{4}.$$
(4.176)

Furthermore, observe that (4.171), and (4.180) imply that for all  $k \in \mathbb{N}_0 \cap [0, M)$  with  $(\mathcal{R}(g_1))(\xi_k) = \frac{g(\xi_k)}{4}$  it holds that

$$(\mathcal{R}(\mathcal{Q}_{1}))(\xi_{k+1}) = 4^{-1} \left[ g(0) + \sum_{j=0}^{k} \alpha_{j} \Re(\xi_{k+1} - \xi_{j}) \right]$$
  
=  $4^{-1} \left[ g(0) + \left[ \sum_{j=0}^{k} \alpha_{j} \Re(\xi_{k} - \xi_{j}) \right] + d \left[ \sum_{j=0}^{k} \alpha_{j} \right] \right]$   
=  $(\mathcal{R}(\mathcal{Q}_{1}))(\xi_{k}) + 4^{-1} [g(\xi_{k+1}) - g(\xi_{k})]$   
=  $\frac{g(\xi_{k+1})}{4}.$  (4.177)

Moreover, note that (4.171), and (4.180) show that for all  $x \in [\xi_M, \infty)$  it holds that

$$(\mathcal{R}(\mathscr{Q}_{1}))(x) = 4^{-1} \left[ g(0) + \sum_{j=0}^{M} \alpha_{j} \Re(x - \xi_{j}) \right]$$
$$= 4^{-1} \left[ g(0) + \left[ \sum_{j=0}^{M} \alpha_{j} \Re(\xi_{M} - \xi_{j}) \right] + (x - \xi_{M}) \left[ \sum_{j=0}^{M} \alpha_{j} \right] \right]$$
$$= (\mathcal{R}(\mathscr{Q}_{1}))(\xi_{M}).$$
(4.178)

Combining this, (4.176), and (4.177) with induction ensures that for all  $k \in \mathbb{N}_0 \cap [0, M]$ ,  $x \in (-\infty, \xi_0], y \in [\xi_M, \infty)$  it holds that

$$(\mathcal{R}(g_1))(\xi_k) = \frac{g(\xi_k)}{4}, \qquad (\mathcal{R}(g_1))(x) = \frac{g(\xi_0)}{4}, \qquad \text{and} \qquad (\mathcal{R}(g_1))(y) = \frac{g(\xi_M)}{4}.$$
 (4.179)

Observe that Proposition 2.10, Proposition 2.7, (4.172), and (4.173) show that for all  $x \in \mathbb{R}$  it holds that

$$(\mathcal{R}(\boldsymbol{\ell}))(x) = (\mathcal{R}(\boldsymbol{g}_2))\big(\mathcal{R}(\boldsymbol{g}_1)(x)\big) = 4(\mathcal{R}(\boldsymbol{g}_1))(x) \tag{4.180}$$

Hence (4.180) demonstrates that for all  $k \in \{1, 2, ..., M\}, x \in [\xi_{k-1}, \xi_k]$  it holds that

$$(\mathcal{R}(\mathcal{L}))(x) = g(0) + \sum_{j=0}^{M} \alpha_j \Re(x - \xi_j) = g(0) + \sum_{j=0}^{k-1} \alpha_j (x - \xi_j).$$
(4.181)

This, (4.179), (4.180), and the fact that  $g(\xi_0) = g(0) = g(\xi_M)$  show that for all  $k \in \{1, 2, ..., M\}$ ,  $x \in [2\pi, \infty)$  it holds that

$$\mathcal{R}(\boldsymbol{\ell})|_{[\xi_{k-1},\xi_k]} \in \mathfrak{L}([\xi_{k-1},\xi_k]) \quad \text{and} \quad (\mathcal{R}(\boldsymbol{\ell}))(x) = g(0) = (\mathcal{R}(\boldsymbol{\ell}))(-x) \quad (4.182)$$

(cf. Definition 3.5). Combining this and (4.179) with (4.180) implies that for all  $k \in \{1, 2, ..., M\}$ ,  $x \in [\xi_{k-1}, \xi_k]$  it holds that

$$(\mathcal{R}(\boldsymbol{f}))(x) = \left(\frac{\xi_k - x}{\xi_k - \xi_{k-1}}\right) (\mathcal{R}(\boldsymbol{f}))(\xi_k) + \left(\frac{x - \xi_{k-1}}{\xi_k - \xi_{k-1}}\right) (\mathcal{R}(\boldsymbol{f}))(\xi_{k-1}) = \left(\frac{\xi_k - x}{\xi_k - \xi_{k-1}}\right) g(\xi_k) + \left(\frac{x - \xi_{k-1}}{\xi_k - \xi_{k-1}}\right) g(\xi_{k-1}).$$

$$(4.183)$$

This ensures that for all  $k \in \{1, 2, ..., M\}$ ,  $x \in [\xi_{k-1}, \xi_k]$  it holds that

$$\begin{aligned} |(\mathcal{R}(\boldsymbol{f}))(x) - g(x)| &= \left| \left( \frac{\xi_{k} - x}{\xi_{k} - \xi_{k-1}} \right) g(\xi_{k}) + \left( \frac{x - \xi_{k-1}}{\xi_{k} - \xi_{k-1}} \right) g(\xi_{k-1}) - g(x) \right| \\ &= \left| \left( \frac{\xi_{k} - x}{\xi_{k} - \xi_{k-1}} \right) (g(\xi_{k}) - g(x)) + \left( \frac{x - \xi_{k-1}}{\xi_{k} - \xi_{k-1}} \right) (g(\xi_{k-1}) - g(x)) \right| \\ &\leq \left( \frac{\xi_{k} - x}{\xi_{k} - \xi_{k-1}} \right) |g(\xi_{k}) - g(x)| + \left( \frac{x - \xi_{k-1}}{\xi_{k} - \xi_{k-1}} \right) |g(\xi_{k-1}) - g(x)| \\ &\leq \left( \frac{\xi_{k} - x}{\xi_{k} - \xi_{k-1}} \right) |\xi_{k} - x| + \left( \frac{x - \xi_{k-1}}{\xi_{k} - \xi_{k-1}} \right) |x - \xi_{k-1}| \\ &\leq |\xi_{k} - \xi_{k-1}| = d \leq \varepsilon \end{aligned}$$

$$(4.184)$$

In addition, note that (4.182) and (4.183) imply that for all  $x, y \in \mathbb{R}$  it holds that

$$|(\mathcal{R}(\boldsymbol{f}))(x) - (\mathcal{R}(\boldsymbol{f}))(y)| \le |x - y| \max_{k \in \{1, 2, \dots, M\}} \frac{g(\xi_k) - g(\xi_{k-1})}{\xi_k - \xi_{k-1}} \le |x - y|.$$
(4.185)

Furthermore, observe that Lemma 2.11, Proposition 2.7, item (II), (4.172), and (4.173) show that

$$\mathcal{D}(\mathbf{\ell}) = (1, 2, M+1, 2, 2, 2, 1) \in \mathbb{N}^{7}.$$
(4.186)

This and the fact that M = 2N + 2 and  $N \ge 2\pi - 1 > 5$  demonstrate that

$$\mathcal{P}(\boldsymbol{\ell}) = \sum_{k=0}^{6} \mathbb{D}_{k}(\boldsymbol{\ell})(\mathbb{D}_{k-1}(\boldsymbol{\ell}) + 1)$$

$$= 2(1+1) + (M+1)(2+1) + 2(M+1+1) + 2(2(2+1)) + 1(2+1)$$

$$= 26 + 5M = 36 + 10N \le 4N^{2}.$$
(4.187)

Moreover, note that Proposition 2.17, item (III), (4.171), (4.172), and (4.173) imply that

$$\mathcal{S}(\boldsymbol{f}) = \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1 \bullet \mathbb{I}_1)\} = \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1), 1\} \le \max\{2, \frac{2\pi}{4}, 2, 1\} = 2.$$
(4.188)

Combining this, (4.174), (4.182), (4.184), and (4.186) with (4.187) establishes items (i), (ii), (iii), (iv), (v), (v), and (vii). The proof of Lemma 4.23 is thus complete.

**Lemma 4.24.** Let  $\varepsilon \in (0,1)$ ,  $n, N \in \mathbb{N}$ ,  $C \in [1,\infty)$  satisfy  $\varepsilon(N+1) \ge 2\pi$  and let  $g: \mathbb{R} \to \mathbb{R}$ satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \le 2$ ,  $g(x+2k\pi) = g(x) \in [-C,C]$ , and  $|g(x) - g(y)| \le |x-y|$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

(i) it holds that  $\mathcal{L}(\not l) \leq n + \log_2(C) + 9$ ,

(ii) it holds it holds that 
$$\mathbb{D}_0(\mathbf{f}) = \mathbb{D}_{\mathcal{L}(\mathbf{f})}(\mathbf{f}) = 1$$
,  $\mathbb{D}_1(\mathbf{f}) \leq 2N$ , and  $\mathbb{D}_{\mathcal{H}(\mathbf{f})}(\mathbf{f}) = 2$ ,

(iii) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(\mathcal{I}))(x) - (\mathcal{R}(\mathcal{I}))(y)| \le |x - y|,$ 

(iv) it holds that  $\sup_{x \in [-2^n \pi, 2^n \pi]} |g(x) - (\mathcal{R}(\boldsymbol{\ell}))(x)| \leq \varepsilon$ ,

- (v) it holds for all  $x \in [2^n \pi, \infty)$  that  $(\mathcal{R}(\mathbf{f}))(-x) = (\mathcal{R}(\mathbf{f}))(x) = g(0)$ ,
- (vi) it holds that  $\mathcal{S}(\mathbf{f}) \leq 2$ , and

(vii) it holds that  $\mathcal{P}(\not{L}) \leq (24 + 18n + 5\log_2(C))N^2$ 

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Lemma 4.24. Throughout this proof assume w.l.o.g. that  $n \geq 2$  (cf. Lemma 4.23), let  $\mathfrak{s} \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $k \in \mathbb{Z}$ ,  $x \in [2k-1, 2k+1)$  that  $\mathfrak{s}(x) = 1 - |x-2k|$ , for every  $j \in \{0, 1, \ldots, N+1\}$  let  $c_j = \frac{2j\pi}{N+1}$ , and for every  $j \in \{1, 2, \ldots, N\}$  let  $f_j \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in [0, 2\pi), y \in (-\infty, -2^n\pi) \cup [2^n\pi, \infty), k \in \mathbb{Z} \cap [-2^{n-1}, 2^{n-1})$  that

$$f(y) = 0 \quad \text{and} \quad f_j(x+2k\pi) = f_j(x) = \begin{cases} g(c_j)\mathfrak{s}\left(\frac{(x-c_{j-1})(N+1)}{2\pi}\right) & : x \in [c_{j-1}, c_{j+1}] \\ 0 & : x \notin [c_{j-1}, c_{j+1}] \end{cases}$$
(4.189)

Observe that Lemma 4.21 (applied with  $n \curvearrowleft n, N \curvearrowleft N, C \curvearrowleft C, \lambda \curvearrowleft g(0), a_j \curvearrowleft c_{j-1}, b_j \curvearrowleft c_{j+1}, c_j \curvearrowleft g(c_j), f_j \curvearrowleft f_j \text{ for } j \in \{1, 2, \ldots, N\}$  in the notation of Lemma 4.21) implies that there exists  $\not \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(\mathcal{I}))(x) = g(0) + \sum_{j=1}^{N} f_j(x)$ ,
- (II) it holds that  $\mathcal{L}(\mathscr{I}) \leq n + \log_2(C) + 9$ ,
- (III) it holds that  $\mathbb{D}_0(\mathcal{L}) = \mathbb{D}_{\mathcal{L}(\mathcal{L})}(\mathcal{L}) = 1$ ,  $\mathbb{D}_1(\mathcal{L}) = 2N$ , and  $\mathbb{D}_{\mathcal{H}(\mathcal{L})}(\mathcal{L}) = 2$ ,
- (IV) it holds that  $\mathcal{P}(\mathcal{L}) \leq (24 + 18n + 5\log_2(C))N^2$ , and
- (V) it holds that  $\mathcal{S}(\not e) \leq 2$

(cf. Definitions 2.1, 2.3, and 2.13). Note that item (I) and Lemma 4.22 (applied with  $\varepsilon \curvearrowleft \varepsilon$ ,  $n \curvearrowleft n, N \curvearrowleft N, c_j \curvearrowleft g(c_j), f_j \curvearrowleft f_j, F \curvearrowleft \mathcal{R}(\mathscr{I})$  for  $j \in \{1, 2, \ldots, N\}$  in the notation of Lemma 4.22) imply that for all  $x \in [-2^n \pi, 2^n \pi], y, z \in \mathbb{R}$  it holds that

$$|g(x) - (\mathcal{R}(\boldsymbol{\ell}))(x)| \le \varepsilon \quad \text{and} \quad |(\mathcal{R}(\boldsymbol{\ell}))(y) - (\mathcal{R}(\boldsymbol{\ell}))(z)| \le |y - z|.$$
(4.190)

Combining this and the fact that  $\mathcal{R}(\not{e}) \in C(\mathbb{R}, \mathbb{R})$  with items (I), (II), (III), (IV), and (V) establishes items (i), (ii), (iii), (iv), (v), (vi), and (vii). The proof of Lemma 4.24 is thus complete.

**Corollary 4.25.** Let  $R \in (0, \infty)$ ,  $\gamma \in (0, 1]$ ,  $\beta \in [1, \infty)$ ,  $\varepsilon \in (0, 1)$  and let  $g: \mathbb{R} \to \mathbb{R}$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq |x - y|$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\mathcal{L}) \in C(\mathbb{R}, \mathbb{R})$ ,
- (*ii*) it holds that  $\sup_{x \in [-R,R]} |g(\gamma \beta x) (\mathcal{R}(\mathcal{L}))(x)| \le \varepsilon$ ,
- (iii) it holds that  $\mathbb{D}_{\mathcal{H}(\ell)}(\ell) = 2$ ,
- (iv) it holds that  $\mathcal{L}(\boldsymbol{\ell}) \leq 16 \max\{1, \lceil \log_2(\beta) \rceil, \lceil \log_2(R) \rceil\},\$
- (v) it holds that  $\mathcal{P}(\mathcal{I}) \leq 4584 \max\{1, \lceil \log_2(R) \rceil, \lceil \log_2(\beta) \rceil\} \varepsilon^{-2}$ , and
- (vi) it holds that  $\mathcal{S}(\mathbf{f}) \leq 2$

(cf. Definition 4.8).

Proof of Corollary 4.25. Throughout this proof assume w.l.o.g. that  $R \geq 2$ . Observe that Lemma 4.24 (applied with  $\varepsilon \curvearrowleft \varepsilon$ ,  $n \curvearrowleft \lceil \log_2(\beta R) \rceil$ ,  $N \curvearrowleft \lceil \frac{2\pi}{\varepsilon} \rceil - 1$ ,  $C \curvearrowleft 6$ ,  $g \backsim g$  in the notation of Lemma 4.24) shows that there exists  $g_2 \in \mathbf{N}$  which satisfies that

(I) it holds for all  $k \in \{0, 1, \dots, \mathcal{L}(g_2)\}$  that  $\mathcal{L}(g_2) \leq \lceil \log_2(\beta R) \rceil + 12$ ,

- (II) it holds it holds that  $\mathbb{D}_0(\mathfrak{g}_2) = \mathbb{D}_{\mathcal{L}(\mathfrak{g}_2)}(\mathfrak{g}_2) = 1$ ,  $\mathbb{D}_1(\mathfrak{g}_2) \leq 14\varepsilon^{-1}$ , and  $\mathbb{D}_{\mathcal{H}(\mathfrak{g}_2)}(\mathfrak{g}_2) = 2$ ,
- (III) it holds that  $\sup_{x \in [-\beta R, \beta R]} |g(x) (\mathcal{R}(g_2))(x)| \le \varepsilon$ ,
- (IV) it holds that  $\mathcal{S}(g_2) \leq 2$ , and
- (V) it holds that  $\mathcal{P}(g_2) \leq (24 + 18\lceil \log_2(\beta R) \rceil + 15) (2\pi\varepsilon^{-1})^2 \leq 2280\lceil \log_2(\beta R) \rceil\varepsilon^{-2}$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8). Note that Corollary 4.6 (applied with  $\beta \curvearrowleft \gamma \beta$ ,  $L \curvearrowleft \lceil \log_2(\gamma \beta) \rceil$  in the notation of Corollary 4.6) demonstrates that there exists  $g_1 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_1))(x) = \gamma \beta x$ ,
- (B) it holds that  $\mathcal{D}(g_1) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{\lceil \log_2(\gamma\beta) \rceil + 2}$ , and
- (C) it holds that  $\mathcal{S}(q_1) \leq 2$ .

Observe that Proposition 2.10, Proposition 2.17, item (II), and item (B) imply that

$$\mathcal{R}(\mathcal{g}_2 \bullet \mathbb{I}_1 \bullet \mathcal{g}_1) = [\mathcal{R}(\mathcal{g}_2)] \circ [\mathcal{R}(\mathcal{g}_1)] \in C(\mathbb{R}, \mathbb{R})$$
(4.191)

(cf. Definitions 2.6 and 2.8). This, item (III), item (A), and the fact that  $\forall x \in [-R, R]: \gamma \beta x \in [-\beta R, \beta R]$  prove that for all  $x \in [-R, R]$  it holds that

$$|g(\gamma\beta x) - (\mathcal{R}(g_2 \bullet \mathbb{I}_1 \bullet g_1))(x)| = |g(\gamma\beta x) - (\mathcal{R}(g_2))(\gamma\beta x)| \le \varepsilon$$
(4.192)

Note that Proposition 2.10, Proposition 2.17, item (I), item (B), and the assumption that  $\gamma \leq 1$  imply that

$$\mathcal{L}(\boldsymbol{g}_{2} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{1}) = \mathcal{L}(\boldsymbol{g}_{2}) + \mathcal{L}(\boldsymbol{g}_{1}) \leq (\lceil \log_{2}(\gamma\beta) \rceil + 1) + (\lceil \log_{2}(\beta R) \rceil + 12) \\ \leq 16 \max\{1, \lceil \log_{2}(\beta) \rceil, \lceil \log_{2}(R) \rceil\}.$$

$$(4.193)$$

This, Lemma 2.11, and item (V) imply that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g_2) + \mathcal{L}(g_1)]$  it holds that

$$\mathbb{D}_{k}(\boldsymbol{g}_{2} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{1}) = \begin{cases} 1 & : k = 0 \\ 2 & : k \in \mathbb{N} \cap (0, \mathcal{L}(\boldsymbol{g}_{1})] \\ \mathbb{D}_{k-\mathcal{L}(\boldsymbol{g}_{1})}(\boldsymbol{g}_{2}) & : k \in \mathbb{N} \cap (\mathcal{L}(\boldsymbol{g}_{1}), \mathcal{L}(\boldsymbol{g}_{2}) + \mathcal{L}(\boldsymbol{g}_{1})]. \end{cases}$$
(4.194)
Hence item (II) and (4.193) show that

$$\begin{aligned} \mathcal{P}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) \\ &= \sum_{k=1}^{\mathcal{L}(g_{2}) + \mathcal{L}(g_{1})} \mathbb{D}_{k}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1})(\mathbb{D}_{k-1}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) + 1) \\ &= 2(1+1) + \left[\sum_{k=2}^{\mathcal{L}(g_{1})} 2(2+1)\right] + \mathbb{D}_{1}(g_{2})(2+1) + \left[\sum_{k=2}^{\mathcal{L}(g_{2})} \mathbb{D}_{k}(g_{2})(\mathbb{D}_{k-1}(g_{2}) + 1)\right] \\ &= 4 + 6(\mathcal{L}(g_{1}) - 1) + \mathbb{D}_{1}(g_{2}) + \left[\sum_{k=1}^{\mathcal{L}(g_{2})} \mathbb{D}_{k}(g_{2})(\mathbb{D}_{k-1}(g_{2}) + 1)\right] \\ &\leq 4 + 6\lceil \log_{2}(\gamma\beta)\rceil + 14\varepsilon^{-1} + \mathcal{P}(g_{2}) \\ &\leq 4 + 6\lceil \log_{2}(\beta)\rceil + 14\varepsilon^{-1} + 2280\lceil \log_{2}(\beta R)\rceil\varepsilon^{-2} \\ &\leq (4 + 6 + 14 + 4560) \max\{1, \lceil \log_{2}(R)\rceil, \lceil \log_{2}(\beta)\rceil\}\varepsilon^{-2} \\ &= 4584 \max\{1, \lceil \log_{2}(R)\rceil, \lceil \log_{2}(\beta)\rceil\}\varepsilon^{-2}. \end{aligned}$$

Furthermore, observe that Proposition 2.17, item (IV), and item (C) demonstrate that

$$\mathcal{S}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) = \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \le \max\{2, 2\} = 2.$$

$$(4.196)$$

This, (4.191), (4.192), (4.193), (4.194), and (4.195) establish items (i), (ii), (iii), (iv), (v), and (vi). The proof of Corollary 4.25 is thus complete.

# 4.5 Upper bounds for approximations of compositions of periodic and product functions

**Lemma 4.26.** Let  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $R \in (0, \infty)$ ,  $\gamma \in (0, 1]$ ,  $\beta \in [1, \infty)$ ,  $g \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq |x - y|$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\mathcal{L}) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (ii) it holds that  $\sup_{x=(x_1,\dots,x_d)\in [-R,R]^d} \left| g\left(\gamma \beta^d \prod_{i=1}^d x_i\right) \mathcal{R}(\mathbf{\ell})(x) \right| \leq \varepsilon$ ,
- (iii) it holds that  $\mathbb{D}_{\mathcal{H}(\mathcal{F})}(\mathcal{F}) = 2$ ,

(iv) it holds that  $\mathcal{L}(\mathcal{L}) \leq 16 \max\{1, \lceil \log_2(R) \rceil, \lceil \log_2(\beta) \rceil\} d^2 \log_2(\varepsilon^{-1}),$ 

- (v) it holds that  $\mathcal{P}(\boldsymbol{\ell}) \leq 12781 \max\{1, \lceil \log_2(R) \rceil, \log_2(\beta)\} d^3 \varepsilon^{-2}$ , and
- (vi) it holds that  $\mathcal{S}(\not l) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Lemma 4.26. Throughout this proof assume w.l.o.g. that  $R \ge 2$  and d > 1 (cf. Corollary 4.25), let  $n, N \in \mathbb{N}$  satisfy  $n = \lceil d \log_2(\beta R) \rceil - 1$  and  $N = \lceil \frac{4\pi}{\varepsilon} \rceil - 1$ . Note that Lemma 4.15 (applied with  $d \curvearrowleft d, \varepsilon \curvearrowleft \frac{\varepsilon}{2}, R \curvearrowleft R, \gamma \curvearrowleft \gamma, \beta \curvearrowleft \beta$  in the notation of Lemma 4.15) ensures that there exists  $g_1 \in \mathbb{N}$  which satisfies that

(I) it holds that  $\mathcal{R}(\mathfrak{g}_1) \in C(\mathbb{R}^d, \mathbb{R})$ ,

(II) it holds that  $\sup_{x=(x_1,\dots,x_d)\in [-R,R]^d} |\gamma\beta^d \prod_{i=1}^d x_i - (\mathcal{R}(g_1))(x)| \leq \frac{\varepsilon}{2}$ ,

(III) it holds that  $\mathcal{L}(g_1) = 8d^2 + 2d^2 \lceil \log_2(R) \rceil + d(\log_2(\varepsilon^{-1}) + 1) + d^2 \lceil \log_2(\beta) \rceil + 2$ ,

- (IV) it holds that that  $\mathbb{D}_1(\mathfrak{g}_1) \leq 2d$  and  $\mathbb{D}_{\mathcal{H}(\mathfrak{g}_1)}(\mathfrak{g}_1) = 2$ ,
- (V) it holds that  $\mathcal{P}(g_1) \leq 8203d^3 + 2048d^3 \lceil \log_2(R) \rceil + 512d^2 (\log_2(\varepsilon^{-1}) + 1) + 514d^3 \log_2(\beta),$ and
- (VI) it holds that  $\mathcal{S}(g_1) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8). Furthermore, observe that the fact that  $\sup_{x \in \mathbb{R}} |g(x)| \leq \lambda + \pi \leq 6$  and Lemma 4.24 (applied with  $\varepsilon \curvearrowleft \frac{\varepsilon}{2}$ ,  $n \curvearrowleft n$ ,  $N \curvearrowleft N$ ,  $C \curvearrowleft 6$ ,  $g \curvearrowleft g$  in the notation of Lemma 4.24) ensure that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (A) it holds that  $\mathcal{L}(g_2) \leq n + \log_2(6) + 9 \leq n + 12$ ,
- (B) it holds it holds that  $\mathbb{D}_0(g_2) = \mathbb{D}_{\mathcal{L}(g_2)}(g_2) = 1$ ,  $\mathbb{D}_1(g_2) \le 2N$ , and  $\mathbb{D}_{\mathcal{H}(g_2)}(g_2) = 2$ ,
- (C) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(g_2))(x) (\mathcal{R}(g_2))(y)| \le |x y|,$
- (D) it holds that  $\sup_{x \in [-2^n \pi, 2^n \pi]} |g(x) (\mathcal{R}(g_2))(x)| \le \frac{\varepsilon}{2}$ ,
- (E) it holds for all  $x \in [2^n \pi, \infty)$  that  $(\mathcal{R}(g_2))(-x) = (\mathcal{R}(g_2))(x) = g(0)$ ,
- (F) it holds that  $\mathcal{S}(g_2) \leq 2$ , and
- (G) it holds that  $\mathcal{P}(g_2) \leq (24 + 18n + 5\log_2(6))N^2 \leq (39 + 18n)N^2$ .

Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{f} = \boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1 \tag{4.197}$$

(cf. Definitions 2.6 and 2.8). Note that (4.197), item (III), item (A), Proposition 2.10, Proposition 2.7, and the fact that  $n \leq d\lceil \log_2(\beta R) \rceil \leq 2d \max\{\lceil \log_2(\beta) \rceil, \lceil \log_2(R) \rceil\}$  show that

$$\begin{aligned} \mathcal{L}(\boldsymbol{f}) \\ &= \mathcal{L}(\boldsymbol{g}_{2}) + \mathcal{L}(\mathbb{I}_{1}) + \mathcal{L}(\boldsymbol{g}_{1}) - 2 \\ &= \mathcal{L}(\boldsymbol{g}_{2}) + \mathcal{L}(\boldsymbol{g}_{1}) \\ &= (n+12) + \left(8d^{2} + 2d^{2}\lceil\log_{2}(R)\rceil + d(\log_{2}(\varepsilon^{-1}) + 1) + d^{2}\lceil\log_{2}(\beta)\rceil + 2\right) \\ &\leq (d\lceil\log_{2}(\beta R)\rceil + 12) + \left(8d^{2} + 2d^{2}\lceil\log_{2}(R)\rceil + d\log_{2}(\varepsilon^{-1}) + d + d^{2}\lceil\log_{2}(\beta)\rceil + 2\right) \quad (4.198) \\ &\leq (d+12 + 8d^{2} + 2d^{2} + d + d + d^{2} + 2) \max\{1, \lceil\log_{2}(R)\rceil, \log_{2}(\varepsilon^{-1}), \lceil\log_{2}(\beta)\rceil\} \\ &= (11d^{2} + 3d + 14) \max\{\lceil\log_{2}(R)\rceil, \log_{2}(\varepsilon^{-1}), \lceil\log_{2}(\beta)\rceil\} \\ &\leq 16d^{2} \max\{\lceil\log_{2}(R)\rceil, \log_{2}(\varepsilon^{-1}), \lceil\log_{2}(\beta)\rceil\} \\ &\leq 16 \max\{\lceil\log_{2}(R)\rceil, \lceil\log_{2}(\beta)\rceil\}d^{2}\log_{2}(\varepsilon^{-1}). \end{aligned}$$

This, (4.197), item (I), item (A), and Lemma 2.11 imply that it holds that  $\mathbb{D}_{\mathcal{H}(\ell)}(\ell) = 2$  and  $\mathcal{D}(\ell)$ 

$$\mathcal{D}(\mathcal{F}) = (\mathbb{D}_0(\mathcal{g}_1), \mathbb{D}_1(\mathcal{g}_1), \dots, \mathbb{D}_{\mathcal{H}(\mathcal{g}_1)}(\mathcal{g}_1), 2\mathbb{D}_{\mathcal{L}(\mathcal{g}_1)}(\mathcal{g}_1), \mathbb{D}_1(\mathcal{g}_2), \mathbb{D}_2(\mathcal{g}_2), \dots, \mathbb{D}_{\mathcal{L}(\mathcal{g}_2)}(\mathcal{g}_2)).$$
(4.199)

Observe that (4.197), Proposition 2.10, and Proposition 2.7 ensure that for all  $x \in \mathbb{R}^d$  it holds that

$$(\mathcal{R}(\boldsymbol{\ell}))(x) = ([\mathcal{R}(\boldsymbol{g}_2)] \circ [\mathcal{R}(\boldsymbol{g}_1)])(x) = (\mathcal{R}(\boldsymbol{g}_2))((\mathcal{R}(\boldsymbol{g}_1))(x)).$$
(4.200)

This, item (II), item (C), item (D), and the fact that for all  $x_1, x_2, \ldots, x_d \in [-R, R]$  it holds that  $|\gamma \beta^d \prod_{i=1}^d x_i| \leq (\beta R)^d \leq 2^{d \log_2(\beta R) - 1} \pi \leq 2^n \pi$  show that for all  $x = (x_1, \ldots, x_d) \in [-R, R]^d$  it holds that

$$\begin{aligned} \left| g\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) - \mathcal{R}(\boldsymbol{f})(\boldsymbol{x}) \right| \\ &= \left| g\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) - (\mathcal{R}(\boldsymbol{g}_{2}))((\mathcal{R}(\boldsymbol{g}_{1}))(\boldsymbol{x})) \right| \end{aligned} \tag{4.201} \\ &= \left| g\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) - (\mathcal{R}(\boldsymbol{g}_{2}))\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) + (\mathcal{R}(\boldsymbol{g}_{2}))\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) - (\mathcal{R}(\boldsymbol{g}_{2}))((\mathcal{R}(\boldsymbol{g}_{1}))(\boldsymbol{x})) \right| \\ &\leq \left| g\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) - (\mathcal{R}(\boldsymbol{g}_{2}))\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) \right| + \left| (\mathcal{R}(\boldsymbol{g}_{2}))\left(\gamma\beta^{d}\prod_{i=1}^{d}x_{i}\right) - (\mathcal{R}(\boldsymbol{g}_{2}))((\mathcal{R}(\boldsymbol{g}_{1}))(\boldsymbol{x})) \right| \\ &\leq \frac{\varepsilon}{2} + \left| \gamma\beta^{d}\prod_{i=1}^{d}x_{i} - (\mathcal{R}(\boldsymbol{g}_{1}))(\boldsymbol{x}) \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Note that (4.199), item (A), item (B), item (IV), and the fact that  $\mathbb{D}_{\mathcal{L}(g_1)}(g_1) = \mathbb{D}_0(g_2)$  demonstrate that

$$\mathcal{P}(\boldsymbol{f}) = \sum_{k=1}^{\mathcal{L}(\boldsymbol{f})} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f}) + 1)$$

$$= \left[\sum_{k=1}^{\mathcal{L}(g_{1})^{-1}} \mathbb{D}_{k}(\boldsymbol{g}_{1})(\mathbb{D}_{k-1}(\boldsymbol{g}_{1}) + 1)\right] + 2\mathbb{D}_{\mathcal{L}(g_{1})}(\boldsymbol{g}_{1})(\mathbb{D}_{\mathcal{L}(g_{1})-1}(\boldsymbol{g}_{1}) + 1)$$

$$+ \mathbb{D}_{1}(\boldsymbol{g}_{2})(2\mathbb{D}_{0}(\boldsymbol{g}_{2}) + 1) + \left[\sum_{k=2}^{\mathcal{L}(g_{2})} \mathbb{D}_{k}(\boldsymbol{g}_{2})(\mathbb{D}_{k-1}(\boldsymbol{g}_{2}) + 1)\right]$$

$$= \left[\sum_{k=1}^{\mathcal{L}(g_{1})} \mathbb{D}_{k}(\boldsymbol{g}_{1})(\mathbb{D}_{k-1}(\boldsymbol{g}_{1}) + 1)\right] + \mathbb{D}_{\mathcal{L}(g_{1})}(\boldsymbol{g}_{1})(\mathbb{D}_{\mathcal{L}(g_{1})-1}(\boldsymbol{g}_{1}) + 1)$$

$$+ \mathbb{D}_{1}(\boldsymbol{g}_{2})\mathbb{D}_{0}(\boldsymbol{g}_{2}) + \left[\sum_{k=1}^{\mathcal{L}(g_{2})} \mathbb{D}_{k}(\boldsymbol{g}_{2})(\mathbb{D}_{k-1}(\boldsymbol{g}_{2}) + 1)\right]$$

$$\leq \mathcal{P}(\boldsymbol{g}_{1}) + 1(2+1) + 2N + \mathcal{P}(\boldsymbol{g}_{2}).$$
(4.202)

Combining this, item (V) and item (G) with the fact that for all  $x \in (0,1)$  it holds that  $-\log_2(x) \leq x^{-2}$  and  $\pi^2 \leq 10$  with the assumption that  $\varepsilon \in (0,1)$ , R > 1,  $d \geq 2$ ,  $n \leq 1$ 

 $\max\{d \log_2(R), 1\}, \text{ and } N \leq \frac{4\pi}{\varepsilon} \text{ proves that}$ 

$$\begin{aligned} \mathcal{P}(\boldsymbol{f}) &\leq \mathcal{P}(\boldsymbol{g}_{1}) + \mathcal{P}(\boldsymbol{g}_{2}) + 3 + 2N \\ &\leq \left(8203d^{3} + 2048d^{3}\lceil \log_{2}(R) \rceil + 512d^{2}(\log_{2}(\varepsilon^{-1}) + 1) + 514d^{3}\log_{2}(\beta)\right) \\ &+ \left((18n + 39)N^{2}\right) + 3 + 2N \\ &\leq 8203d^{3} + 2048d^{3}\lceil \log_{2}(R) \rceil + 512d^{2}\log_{2}(\varepsilon^{-1}) + 512d^{2} + 514d^{3}\log_{2}(\beta) \\ &+ (18\max\{d\log_{2}(\beta R), 1\} + 39)16\pi^{2}\varepsilon^{-2} + 3 + 8\pi\varepsilon^{-1} \\ &\leq 8203d^{3} + 2048d^{3}\lceil \log_{2}(R) \rceil + 512d^{2}\log_{2}(\varepsilon^{-1}) + 512d^{2} + 514d^{3}\log_{2}(\beta) \\ &+ (36\max\{\log_{2}(R), \log_{2}(\beta), 1\}d + 39)160\varepsilon^{-2} + 3 + 24\varepsilon^{-1} \\ &= 8203d^{3} + 2048d^{3}\lceil \log_{2}(R) \rceil + 512d^{2}\log_{2}(\varepsilon^{-1}) + 512d^{2} + 514d^{3}\log_{2}(\beta) \\ &+ 2880\max\{\log_{2}(R), \log_{2}(\beta)\}d\varepsilon^{-2} + 6240\varepsilon^{-2} + 3 + 24\varepsilon^{-1} \\ &= (8203d^{3} + 2048d^{3} + 512d^{2} + 512d^{2} + 514d^{3} \\ &+ 2880d + 6240 + 3 + 24\right)\max\{\lceil \log_{2}(R) \rceil, \log_{2}(\beta)\}\varepsilon^{-2} \\ &\leq (10765d^{3} + 1024d^{2} + 2880d + 6267)\max\{\lceil \log_{2}(R) \rceil, \log_{2}(\beta)\}\varepsilon^{-2} \\ &\leq (10765 + 512 + 720 + 784)d^{3}\max\{\lceil \log_{2}(R) \rceil, \log_{2}(\beta)\}\varepsilon^{-2} \\ &\leq 12781d^{3}\max\{\lceil \log_{2}(R) \rceil, \log_{2}(\beta)\}\varepsilon^{-2}. \end{aligned}$$

Moreover, observe that item (VI), item (F), and Proposition 2.17 show that

$$\mathcal{S}(\boldsymbol{\ell}) = \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \le \max\{2, 4\} = 4.$$
(4.204)

Combining this, (4.199), and (4.203) with (4.201) establishes items (i), (ii), (iii), (iv), (v), and (vi). The proof of Lemma 4.26 is thus complete.

**Lemma 4.27.** Let  $d \in \mathbb{N}$ ,  $\kappa, R \in (0, \infty)$ ,  $\varepsilon \in (0, \kappa)$ ,  $\gamma \in (0, 1]$ ,  $\beta \in [1, \infty)$   $g \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2\kappa$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq \kappa |x - y|$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\mathscr{L}) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (ii) it holds that  $\sup_{x=(x_1,\dots,x_d)\in [-R,R]^d} |g(\gamma\beta^d\prod_{i=1}^d x_i) \mathcal{R}(\boldsymbol{\ell})(x)| \leq \varepsilon$ ,
- (iii) it holds that  $\mathbb{D}_{\mathcal{H}(\mathcal{F})}(\mathcal{F}) = 2$ ,

(iv) it holds that  $\mathcal{P}(\mathcal{I}) \leq 12802 \max\{1, \lceil \log_2(R) \rceil, \log_2(\beta)\} \max\{1, \kappa^3\} d^3 \varepsilon^{-2}$ ,

- (v) it holds that  $\mathcal{L}(\mathbf{\ell}) \leq 19 \max\{1, \lceil \log_2(R) \rceil, \lceil \log_2(\beta) \rceil\} \max\{1, \kappa^2\} d^2 \varepsilon^{-1}$ , and
- (vi) it holds that  $\mathcal{S}(\not e) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Lemma 4.27. Throughout this proof assume w.l.o.g. that  $\kappa \geq 2$ , let  $L \in \mathbb{N}$  satisfy  $L = \lceil \log_2(\kappa) \rceil$ , and let  $f \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x \in \mathbb{R}$  that  $f(x) = \kappa^{-1}g(x)$ . Note that Lemma 4.26 (applied with  $d \curvearrowleft d, \varepsilon \curvearrowleft \frac{\varepsilon}{\kappa}, R \curvearrowleft R, \gamma \curvearrowleft \gamma, \beta \curvearrowleft \beta, g \backsim f$  in the notation of Lemma 4.26) demonstrates that there exists  $g_1 \in \mathbb{N}$  which satisfies that

(I) it holds that  $\mathcal{R}(\mathfrak{g}_1) \in C(\mathbb{R}^d, \mathbb{R})$ ,

(II) it holds that  $\sup_{x=(x_1,\dots,x_d)\in [-R,R]^d} \left| f\left(\gamma \beta^d \prod_{i=1}^d x_i\right) - \mathcal{R}(g_1)(x) \right| \leq \frac{\varepsilon}{\kappa}$ ,

(III) it holds that  $\mathbb{D}_{\mathcal{H}(g_1)}(g_1) = 2$ ,

(IV) it holds that  $\mathcal{L}(q_1) \leq 16 \max\{1, \lceil \log_2(R) \rceil, \lceil \log_2(\beta) \rceil\} d^2 \log_2(\kappa \varepsilon^{-1}),$ 

- (V) it holds that  $\mathcal{P}(g_1) \leq 12781 \max\{1, \lceil \log_2(R) \rceil, \log_2(\beta)\} d^3 \kappa^2 \varepsilon^{-2}$ , and
- (VI) it holds that  $\mathcal{S}(g_1) \leq 4$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8). Nobs that Corollary 4.6 (applied with  $\beta \curvearrowleft \kappa$ ,  $L \curvearrowleft L$  in the notation of Corollary 4.6) shows that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_2))(x) = \kappa x$ ,
- (B) it holds that  $\mathcal{D}(g_2) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{L+2}$ , and
- (C) it holds that  $\mathcal{S}(q_2) \leq 2$ .

Observe that item (II), item (A), Proposition 2.10, and Proposition 2.7 demonstrate that for all  $x = (x_1, \ldots, x_d) \in [-R, R]^d$  it holds that  $\mathcal{R}(g_2 \bullet \mathbb{I}_1 \bullet g_1) \in C(\mathbb{R}^d, \mathbb{R})$  and

$$|(\mathcal{R}(\mathcal{g}_{2} \bullet \mathbb{I}_{1} \bullet \mathcal{g}_{1}))(x) - g(\gamma \beta^{d} \prod_{i=1}^{d} x_{i})| = |([\mathcal{R}(\mathcal{g}_{2})] \circ [\mathcal{R}(\mathcal{g}_{1})])(x) - g(\gamma \beta^{d} \prod_{i=1}^{d} x_{i})|$$
  
=  $\kappa |(\mathcal{R}(\mathcal{g}_{1}))(x) - f(\gamma \beta^{d} \prod_{i=1}^{d} x_{i})| \le \varepsilon.$  (4.205)

Note that item (IV), item (B), Proposition 2.10, and Proposition 2.7 imply

$$\mathcal{L}(\boldsymbol{g}_{2} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{1}) = \mathcal{L}(\boldsymbol{g}_{2}) + 2 + \mathcal{L}(\boldsymbol{g}_{1}) - 2$$

$$= \mathcal{L}(\boldsymbol{g}_{1}) + L + 1$$

$$\leq 16 \max\{1, \lceil \log_{2}(R) \rceil, \lceil \log_{2}(\beta) \rceil\} d^{2} \log_{2}(\kappa \varepsilon^{-1}) + \lceil \log_{2}(\kappa) \rceil + 1$$

$$\leq 16 \max\{1, \lceil \log_{2}(R) \rceil, \lceil \log_{2}(\beta) \rceil\} d^{2} \kappa \varepsilon^{-1} + \kappa + 2$$

$$\leq 16 \max\{1, \lceil \log_{2}(R) \rceil, \lceil \log_{2}(\beta) \rceil\} d^{2} \kappa \varepsilon^{-1} + \kappa^{2} \varepsilon^{-1} + 2\kappa \varepsilon^{-1}$$

$$\leq 19 \max\{1, \lceil \log_{2}(R) \rceil, \lceil \log_{2}(\beta) \rceil\} \max\{1, \kappa^{2}\} d^{2} \varepsilon^{-1}$$

$$(4.206)$$

Observe that item (I), item (IV), item (B), and Proposition 2.10 hence imply that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g_2 \bullet \mathbb{I}_1 \bullet g_1)]$  it holds that

$$\mathbb{D}_{k}(\boldsymbol{g}_{2} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{1}) = \begin{cases} \mathbb{D}_{k}(\boldsymbol{g}_{1}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\boldsymbol{g}_{1})) \\ 2 & : k \in \mathbb{N} \cap [\mathcal{L}(\boldsymbol{g}_{1}), \mathcal{L}(\boldsymbol{g}_{1}) + L + 1) \\ 1 & : k = \mathcal{L}(\boldsymbol{g}_{1}) + L + 1. \end{cases}$$
(4.207)

This and item (III) show that

$$\mathcal{P}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) = \sum_{k=1}^{\mathcal{L}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1})} \mathbb{D}_{k}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1})(\mathbb{D}_{k-1}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) + 1) \\ = \left[\sum_{k=1}^{\mathcal{L}(g_{1})-1} \mathbb{D}_{k}(g_{1})(\mathbb{D}_{k-1}(g_{1}) + 1)\right] + \left[\sum_{k=\mathcal{L}(g_{1})}^{\mathcal{L}(g_{1})+L} 2(2+1)\right] + 1(2+1) \\ = \mathcal{P}(g_{1}) + 6(L+1) + 3 \\ \leq 12781 \max\{1, \lceil \log_{2}(R) \rceil, \log_{2}(\beta)\} d^{3}\kappa^{2}\varepsilon^{-2} + 6(\lceil \log_{2}(\kappa) \rceil + 1) + 3 \\ \leq 12796 \max\{1, \lceil \log_{2}(R) \rceil, \log_{2}(\beta)\} d^{3}\kappa^{2}\varepsilon^{-2} + 6\kappa \\ \leq 12796 \max\{1, \lceil \log_{2}(R) \rceil, \log_{2}(\beta)\} d^{3}\kappa^{2}\varepsilon^{-2} + 6\kappa^{3}\varepsilon^{-2} \\ \leq 12802 \max\{1, \lceil \log_{2}(R) \rceil, \log_{2}(\beta)\} \max\{\kappa^{3}, 1\} d^{3}\varepsilon^{-2}.$$

Furthermore, note that item (VI), item (C), and Proposition 2.17 demonstrate that

$$\mathcal{S}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) = \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} = \max\{2, 4\} = 4.$$
(4.209)

Combining this, (4.206), (4.207), and (4.208) establishes items (i), (ii), (iii), (iv), (v), and (vi). The proof of Lemma 4.27 is thus complete.

**Corollary 4.28.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d \in \mathbb{N}$ ,  $\kappa, R, \mathfrak{c} \in (0, \infty)$ ,  $\varepsilon \in (0, \kappa)$ ,  $\gamma \in (0, 1]$ ,  $\beta \in [1, \infty)$  satisfy  $R = \lceil \log_2(\max\{2, |a|, |b|, \beta\}) \rceil$  and  $\mathfrak{c} \geq 13968R \max\{1, \kappa^3\}$ , let  $g \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2\kappa$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq \kappa |x - y|$ , and let  $f \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in [a, b]^d$  that  $f(x) = g(\gamma \beta^d \prod_{i=1}^d x_i)$  (cf. Definition 4.8). Then

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not f \in \mathbf{N} : (\mathcal{P}(\not f) = p) \land (\mathcal{L}(\not f) \leq \mathfrak{c} d^2 \varepsilon^{-1}) \land \\ (\mathcal{S}(\not f) \leq 1) \land (\mathcal{R}(\not f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not f))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c} d^3 \varepsilon^{-2} \quad (4.210)$$

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Corollary 4.28. Throughout this proof assume w.l.o.g. that  $\max\{|a|, |b|\} > 0$  and let  $c \in [1, \infty)$  satisfy  $c = R \max\{1, \kappa^3\}$ . Observe that Lemma 4.27 (applied with  $d \curvearrowleft d, R \backsim \max\{|a|, |b|\}, \kappa \backsim \kappa, \varepsilon \backsim \varepsilon, \gamma \backsim \gamma$  in the notation of Lemma 4.27) shows that there exists  $g \in \mathbf{N}$  which satisfies that

- (i) it holds that  $\mathcal{R}(q) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (ii) it holds that  $\sup_{x \in [a,b]^d} |(\mathcal{R}(g))(x) f(x)| \le \varepsilon$ ,
- (iii) it holds that  $\mathbb{D}_{\mathcal{H}(g)}(g) = 2$ ,
- (iv) it holds that  $\mathcal{L}(q) \leq 19cd^2\varepsilon^{-1}$ ,
- (v) it holds that  $\mathcal{P}(g) \leq 12802cd^3\varepsilon^{-2}$ , and
- (vi) it holds that  $\mathcal{S}(q) \leq 4$

(cf. Definitions 2.1, 2.3, and 2.13). Note that Lemma 4.3 (applied with  $d \curvearrowleft d$ ,  $\not f \curvearrowleft g$  in the notation of Lemma 4.3) hence demonstrates that there exists  $\not h \in \mathbf{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(\mathscr{h}) \in C(\mathbb{R}^d, \mathbb{R}),$
- (II) it holds that  $\sup_{x \in [a,b]^d} |(\mathcal{R}(\mathscr{H}))(x) f(x)| \leq \varepsilon$ ,
- (III) it holds that  $\mathbb{D}_{\mathcal{H}(\mathscr{R})}(\mathscr{R}) = 4$ ,
- (IV) it holds that  $\mathcal{L}(\hbar) \leq 2(19cd^2\varepsilon^{-1}) + 1 \leq 39cd^2\varepsilon^{-1}$ ,
- (V) it holds that  $\mathcal{P}(\mathbb{A}) \leq 12802cd^3\varepsilon^{-2} + 2 + 20(19cd^2\varepsilon^{-1}) \leq 13184cd^3\varepsilon^{-2}$ , and
- (VI) it holds that  $\mathcal{S}(\mathscr{R}) \leq 2$ .

Observe that Lemma 4.3 (applied with  $d \curvearrowleft d$ ,  $\not {e} \curvearrowleft \not {k}$  in the notation of Lemma 4.3) therefore implies that there exists  $\not {e} \in \mathbb{N}$  which satisfies that

- (A) it holds that  $\mathcal{R}(\mathscr{L}) \in C(\mathbb{R}^d, \mathbb{R}),$
- (B) it holds that  $\sup_{x \in [a,b]^d} |(\mathcal{R}(\mathbf{f}))(x) f(x)| \leq \varepsilon$ ,
- (C) it holds that  $\mathcal{L}(\mathbf{f}) \leq 2(39cd^2\varepsilon^{-1}) + 1 \leq 79cd^2\varepsilon^{-1}$ ,
- (D) it holds that  $\mathcal{P}(\mathbf{p}) \leq 13184cd^3\varepsilon^{-2} + 4 + 20(39cd^2\varepsilon^{-1}) \leq 13968cd^3\varepsilon^{-2}$ , and
- (E) it holds that  $\mathcal{S}(\not l) \leq 1$ .

Hence we obtain (4.210). The proof of Corollary 4.28 is thus complete.

# 4.6 Upper bounds for approximations of certain smooth and bounded functions

**Lemma 4.29.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d \in \mathbb{N}$ ,  $\gamma \in (0, \infty)$ ,  $\varepsilon \in (0, 1)$ ,  $g \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq |x - y|$ . Then there exists  $\mathfrak{f} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\not e) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (ii) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left| g\left(\gamma 2^d \sum_{i=1}^d x_i\right) (\mathcal{R}(\mathbf{f}))(x) \right| \leq \varepsilon$ ,
- (iii) it holds that  $\mathbb{D}_1(\mathcal{L}) = \mathbb{D}_{\mathcal{H}(\mathcal{L})}(\mathcal{L}) = 2$ ,
- (iv) it holds that  $\mathcal{L}(\mathcal{L}) \leq 15d \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil$ ,
- (v) it holds that  $\mathcal{P}(\mathcal{L}) \leq 2304 \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil d\varepsilon^{-2}$ , and
- (vi) it holds that  $\mathcal{S}(\not l) \leq 2$
- (cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Lemma 4.29. Throughout this proof assume w.l.o.g. that d > 1 (cf. Corollary 4.25), let  $n, N \in \mathbb{N}$  satisfy  $n = 2d \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil$  and  $N = \lceil \frac{2\pi}{\varepsilon} \rceil - 1$ , and let  $g_1 \in \mathbb{N}$  satisfy

$$g_1 = \left( \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}, 0 \right) \in \left( \mathbb{R}^{1 \times d} \times \mathbb{R}^1 \right)$$

$$(4.211)$$

(cf. Definitions 2.1 and 4.8). Note that Corollary 4.6 (applied with  $\beta \curvearrowleft \gamma 2^d$ ,  $L \curvearrowleft d + \lceil \log_2(\max\{1,\gamma\}) \rceil$  in the notation of Corollary 4.6) shows that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_2))(x) = \gamma 2^d x$ ,
- (II) it holds that  $\mathcal{D}(g_2) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{d + \lceil \log_2(\max\{1, \gamma\}) \rceil + 2}$ ,
- (III) it holds that  $\mathbb{S}_0(\mathfrak{g}_2) = 1$ ,  $\mathbb{S}_1(\mathfrak{g}_2) = 2$ , and  $\mathcal{S}(\mathfrak{g}_2) = 2$ , and
- (IV) it holds that  $\mathcal{P}(g_2) = 6(d + \lceil \log_2(\max\{1, \gamma\}) \rceil) + 1.$

(cf. Definitions 2.3 and 2.13). Furthermore, observe that the fact that  $\sup_{x \in \mathbb{R}} |g(x)| \leq \lambda + \pi \leq 6$ and Lemma 4.24 (applied with  $\varepsilon \curvearrowleft \varepsilon$ ,  $n \curvearrowleft n$ ,  $N \curvearrowleft N$ ,  $C \curvearrowleft 6$ ,  $g \backsim g$  in the notation of Lemma 4.24) ensure that there exists  $g_3 \in \mathbf{N}$  which satisfies that

- (A) it holds that  $\mathcal{L}(g_3) \leq n + \log_2(6) + 9 \leq n + 12$  and
- (B) it holds it holds that  $\mathbb{D}_0(\mathfrak{g}_3) = \mathbb{D}_{\mathcal{L}(\mathfrak{g}_3)}(\mathfrak{g}_3) = 1$ ,  $\mathbb{D}_1(\mathfrak{g}_3) \leq 2N$ , and  $\mathbb{D}_{\mathcal{H}(\mathfrak{g}_3)}(\mathfrak{g}_3) = 2$ ,
- (C) it holds for all  $x, y \in \mathbb{R}$  that  $|(\mathcal{R}(g_3))(x) (\mathcal{R}(g_3))(y)| \le |x y|,$
- (D) it holds that  $\sup_{x \in [-2^n \pi, 2^n \pi]} |g(x) (\mathcal{R}(g_3))(x)| \le \varepsilon$ ,
- (E) it holds for all  $x \in [2^n \pi, \infty)$  that  $(\mathcal{R}(g_3))(-x) = (\mathcal{R}(g_3))(x) = g(0)$ ,
- (F) it holds that  $\mathcal{S}(g_3) \leq 2$ , and
- (G) it holds that  $\mathcal{P}(g_3) \leq (24 + 18n + 5\log_2(6))N^2 \leq (18n + 39)N^2$ .

Next let  $\not \in \mathbf{N}$  satisfy

$$\boldsymbol{\ell} = \boldsymbol{g}_3 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_2 \bullet \boldsymbol{g}_1 \tag{4.212}$$

(cf. Definitions 2.6 and 2.8). Note that (4.211), (4.212), item (II), item (A), Proposition 2.10, and Proposition 2.7 show that

$$\mathcal{L}(\boldsymbol{f}) = \mathcal{L}(\boldsymbol{g}_{3}) + \mathcal{L}(\mathbb{I}_{1}) + \mathcal{L}(\boldsymbol{g}_{2}) + \mathcal{L}(\boldsymbol{g}_{1}) - 3 
= \mathcal{L}(\boldsymbol{g}_{3}) + \mathcal{L}(\boldsymbol{g}_{2}) 
= (n+12) + (d + \lceil \log_{2}(\max\{1,\gamma\}) \rceil + 1) 
= d + 2d \lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil + \lceil \log_{2}(\max\{1,\gamma\}) \rceil + 13 
\leq (3d + 1 + 13) \lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil 
\leq 10d \lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil.$$
(4.213)

Observe that (4.197), (4.211), item (I), Proposition 2.10, and Proposition 2.7 ensure that for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  it holds that

$$(\mathcal{R}(\boldsymbol{\ell}))(x) = ([\mathcal{R}(\boldsymbol{g}_3)] \circ [\mathcal{R}(\boldsymbol{g}_2)] \circ [\mathcal{R}(\boldsymbol{g}_1)])(x) = ([\mathcal{R}(\boldsymbol{g}_3)] \circ [\mathcal{R}(\boldsymbol{g}_2)]) \left(\sum_{i=1}^d x_i\right)$$

$$= (\mathcal{R}(\boldsymbol{g}_3)) \left(\gamma 2^d \sum_{i=1}^d x_i\right).$$
(4.214)

This, item (D), and the fact that for all  $x_1, x_2, \ldots, x_d \in [a, b]$  it holds that  $|\gamma 2^d \sum_{i=1}^d x_i| \leq \gamma d2^d \max\{2, |a|, |b|\} \leq 2^{2d \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil} \pi = 2^n \pi$  imply that for all  $x = (x_1, \ldots, x_d) \in [a, b]^d$  it holds that

$$\left|g\left(\gamma 2^d \sum_{i=1}^d x_i\right) - (\mathcal{R}(\mathscr{f}))(x)\right| = \left|g\left(\gamma 2^d \sum_{i=1}^d x_i\right) - (\mathcal{R}(\mathscr{G}_3))\left(2^d \sum_{i=1}^d x_i\right)\right| \le \varepsilon.$$
(4.215)

Moreover, note that (4.211), (4.212), (4.213), item (I), item (A), and Lemma 2.11 imply that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(\mathcal{L})]$  it holds that

$$\mathbb{D}_{k}(\boldsymbol{\ell}) = \begin{cases} d & : k = 0 \\ \mathbb{D}_{k}(\boldsymbol{g}_{2}) & : k \in \mathbb{N} \cap (0, \mathcal{L}(\boldsymbol{g}_{2})) \\ 2 & : k = \mathcal{L}(\boldsymbol{g}_{2}) \\ \mathbb{D}_{k-\mathcal{L}(\boldsymbol{g}_{2})}(\boldsymbol{g}_{3}) & : k \in \mathbb{N} \cap (\mathcal{L}(\boldsymbol{g}_{2}), \mathcal{L}(\boldsymbol{\ell})]. \end{cases}$$
(4.216)

This, item (II), and item (B) ensure that

$$\mathcal{P}(\boldsymbol{f}) = \sum_{k=1}^{\mathcal{L}(\boldsymbol{f})} \mathbb{D}_{k}(\boldsymbol{f})(\mathbb{D}_{k-1}(\boldsymbol{f}) + 1)$$

$$= \mathbb{D}_{1}(\boldsymbol{g}_{2})(d+1) + \left[\sum_{k=2}^{\mathcal{L}(\boldsymbol{g}_{2})-1} \mathbb{D}_{k}(\boldsymbol{g}_{2})(\mathbb{D}_{k-1}(\boldsymbol{g}_{2}) + 1)\right] + 2(\mathbb{D}_{\mathcal{L}(\boldsymbol{g}_{2})-1}(\boldsymbol{g}_{2}) + 1)$$

$$+ \mathbb{D}_{1}(\boldsymbol{g}_{3})(2+1) + \left[\sum_{k=2}^{\mathcal{L}(\boldsymbol{g}_{3})} \mathbb{D}_{k}(\boldsymbol{g}_{3})(\mathbb{D}_{k-1}(\boldsymbol{g}_{3}) + 1)\right]$$

$$= \mathbb{D}_{1}(\boldsymbol{g}_{2})(d-1) + \left[\sum_{k=1}^{\mathcal{L}(\boldsymbol{g}_{2})} \mathbb{D}_{k}(\boldsymbol{g}_{2})(\mathbb{D}_{k-1}(\boldsymbol{g}_{2}) + 1)\right] + \mathbb{D}_{\mathcal{L}(\boldsymbol{g}_{2})-1}(\boldsymbol{g}_{2}) + 1$$

$$+ \mathbb{D}_{1}(\boldsymbol{g}_{3}) + \left[\sum_{k=1}^{\mathcal{L}(\boldsymbol{g}_{3})} \mathbb{D}_{k}(\boldsymbol{g}_{3})(\mathbb{D}_{k-1}(\boldsymbol{g}_{3}) + 1)\right]$$

$$\leq 2d-2 + \mathcal{P}(\boldsymbol{g}_{2}) + 3 + 2N + \mathcal{P}(\boldsymbol{g}_{3}) = \mathcal{P}(\boldsymbol{g}_{2}) + \mathcal{P}(\boldsymbol{g}_{3}) + 2N + 2d + 1.$$
(4.217)

Combining this, item (IV), and item (G) with the fact that for all  $x \in (0,1)$  it holds that  $-\log_2(x) \le x^{-2}$  and  $\pi^2 \le 10$  with the assumption that  $n = 2d \lceil \log_2(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil$ ,

 $\varepsilon \in (0,1), d \ge 2$ , and  $N \le \frac{2\pi}{\varepsilon}$  demonstrates that

$$\begin{aligned} \mathcal{P}(\boldsymbol{f}) \\ &\leq \mathcal{P}(\boldsymbol{g}_{2}) + \mathcal{P}(\boldsymbol{g}_{3}) + 2N + 2d + 1 \\ &\leq (6(d + \lceil \log_{2}(\max\{1,\gamma\}) \rceil) + 1) + (18n + 39)N^{2} + 2N + 2d + 1 \\ &\leq (18n + 39)4\pi^{2}\varepsilon^{-2} + 6\lceil \log_{2}(\max\{1,\gamma\}) \rceil + 4\pi\varepsilon^{-1} + 8d + 2 \\ &\leq ((18n + 39)40 + 6\lceil \log_{2}(\max\{1,\gamma\}) \rceil + 13 + 8d + 2)\varepsilon^{-2} \\ &= (1440d\lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil + 6\lceil \log_{2}(\max\{1,\gamma\}) \rceil + 8d + 1575)\varepsilon^{-2} \\ &\leq (1440d\lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil + 3d\lceil \log_{2}(\max\{1,\gamma\}) \rceil + 8d + 788d)\varepsilon^{-2} \\ &\leq (1440 + 3 + 8 + 788)\lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil d\varepsilon^{-2} \\ &= 2239\lceil \log_{2}(\max\{1,\gamma\}\max\{2,|a|,|b|\}) \rceil d\varepsilon^{-2}. \end{aligned}$$

In addition, observe that (4.211), (4.212), item (III), item (F), Lemma 2.16, and Proposition 2.17 show that

$$\mathcal{S}(\boldsymbol{f}) = \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\boldsymbol{g}_2 \bullet \boldsymbol{g}_1)\} \leq \max\{\mathcal{S}(\boldsymbol{g}_3), \mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1), \mathbb{S}_0(\boldsymbol{g}_2)(\mathbb{S}_1(\boldsymbol{g}_1)+1)\} \\ \leq \max\{2, 2, 1, 1(1+1)\} = 2.$$

$$(4.219)$$

Combining this with (4.213), (4.216), (4.218), and (4.215) establishes items (i), (ii), (iii), (iv), (v), and (vi). The proof of Lemma 4.29 is thus complete.

**Corollary 4.30.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d \in \mathbb{N}$ ,  $\gamma, \kappa \in (0, \infty)$ ,  $\varepsilon \in (0, \kappa)$ ,  $g \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2\kappa$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq \kappa |x - y|$ . Then there exists  $\not{e} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\not e) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (ii) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left| g\left(\gamma 2^d \sum_{i=1}^d x_i\right) (\mathcal{R}(\mathscr{I}))(x) \right| \leq \varepsilon$ ,
- (iii) it holds that  $\mathbb{D}_{\mathcal{H}(f)}(f) = 2$ ,
- (iv) it holds that  $\mathcal{L}(\mathcal{L}) \leq 22 \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil \max\{\kappa, 1\} d$ ,
- (v) it holds that  $\mathcal{P}(\mathcal{L}) \leq 2316 \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil \max\{\kappa^3, 1\}d$ , and
- (vi) it holds that  $\mathcal{S}(\mathbf{f}) \leq 2$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Corollary 4.30. Throughout this proof let  $n, R \in \mathbb{N}$ ,  $f \in C(\mathbb{R}, \mathbb{R})$  satisfy for all  $x \in \mathbb{R}$ that  $n = \lceil \log_2(\max\{\kappa, 2\}) \rceil$ ,  $R = \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil$ , and  $f(x) = \kappa^{-1}g(x)$  (cf. Definition 4.8). Note that Lemma 4.29 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, \gamma \curvearrowleft \gamma, \varepsilon \curvearrowleft \frac{\varepsilon}{\kappa}$ ,  $g \curvearrowleft f$  in the notation of Lemma 4.29) shows that there exists  $g_1 \in \mathbb{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(g_1) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (II) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left| f\left(\gamma 2^d \sum_{i=1}^d x_i\right) (\mathcal{R}(\mathcal{G}_1))(x) \right| \leq \frac{\varepsilon}{\kappa}$ ,

- (III) it holds that  $\mathbb{D}_1(g_1) = \mathbb{D}_{\mathcal{H}(g_1)}(g_1) = 2$ ,
- (IV) it holds that  $\mathcal{L}(g_1) \leq 15Rd$ ,
- (V) it holds that  $\mathcal{P}(g_1) \leq 2304R\kappa^2 d\varepsilon^{-2}$ , and
- (VI) it holds that  $\mathcal{S}(g_1) \leq 2$

(cf. Definitions 2.1, 2.3, and 2.13). Observe that Corollary 4.6 (applied with  $\beta \curvearrowleft \kappa$ ,  $L \backsim n$  in the notation of Corollary 4.6) and the fact that  $\kappa \leq 2^n$  show that there exists  $g_2 \in \mathbf{N}$  which satisfies that

- (A) it holds for all  $x \in \mathbb{R}$  that  $(\mathcal{R}(g_2))(x) = \kappa x$ ,
- (B) it holds that  $\mathcal{D}(g_2) = (1, 2, 2, \dots, 2, 1) \in \mathbb{N}^{n+2}$ ,
- (C) it holds that  $\mathbb{S}_0(\mathfrak{g}_2) = 1$  and  $\mathcal{S}(\mathfrak{g}_2) = \max\{1, \kappa^{\frac{1}{n}}\} \leq 2$ , and
- (D) it holds that  $\mathcal{P}(\mathfrak{g}_2) = 6n + 1$ .

Note that item (I), item (B), Proposition 2.10, and Proposition 2.7 ensure that

$$\mathcal{R}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) \in C(\mathbb{R}^d, \mathbb{R}) \quad \text{and} \quad \mathcal{L}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) = \mathcal{L}(\boldsymbol{g}_2) + \mathcal{L}(\boldsymbol{g}_1) \quad (4.220)$$

(cf. Definitions 2.6 and 2.8). Combining this with item (IV), item (B), and the fact that for all  $x \in [2, \infty)$  it holds that  $\lceil \log_2(x) \rceil \leq \log_2(x) + 1 \leq x$  implies that

$$\mathcal{L}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) = \mathcal{L}(\boldsymbol{g}_2) + \mathcal{L}(\boldsymbol{g}_1) \le 15Rd + 6\lceil \log_2(\max\{\kappa, 2\}) \rceil + 1 \le 22R \max\{\kappa, 1\}d.$$
(4.221)

Furthermore, observe that item (II), item (A), Proposition 2.10, and Proposition 2.7 demonstrate that for all  $x = (x_1, \ldots, x_d) \in [a, b]^d$  it holds that

$$\left|g\left(\gamma 2^d \sum_{i=1}^d x_i\right) - \left(\mathcal{R}(\mathcal{Q}_2 \bullet \mathbb{I}_1 \bullet \mathcal{Q}_1)\right)(x)\right| = \kappa \left|f\left(\gamma 2^d \sum_{i=1}^d x_i\right) - \left(\mathcal{R}(\mathcal{Q}_1)\right)(x)\right| \le \varepsilon.$$
(4.222)

Moreover, note that item (VI), item (C), Proposition 2.17, and the fact that  $\kappa^{\frac{1}{n}} \leq 2$  show that

$$\mathcal{S}(\boldsymbol{g}_2 \bullet \mathbb{I}_1 \bullet \boldsymbol{g}_1) \le \max\{\mathcal{S}(\boldsymbol{g}_2), \mathcal{S}(\boldsymbol{g}_1)\} \le \max\{\kappa^{\frac{1}{n}}, 2\} = 2.$$
(4.223)

In addition, observe that (4.221), item (B), Lemma 2.11, and Proposition 2.7 imply that for all  $k \in \mathbb{N}_0 \cap [0, \mathcal{L}(g_2 \bullet \mathbb{I}_1 \bullet g_1)]$  it holds that

$$\mathbb{D}_{k}(\boldsymbol{g}_{2} \bullet \mathbb{I}_{1} \bullet \boldsymbol{g}_{1}) = \begin{cases} \mathbb{D}_{k}(\boldsymbol{g}_{1}) & : k \in \mathbb{N}_{0} \cap [0, \mathcal{L}(\boldsymbol{g}_{1})) \\ 2 & : k \in \mathbb{N} \cap [\mathcal{L}(\boldsymbol{g}_{1}), \mathcal{L}(\boldsymbol{g}_{1}) + \mathcal{L}(\boldsymbol{g}_{2})) \\ 1 & : k = \mathcal{L}(\boldsymbol{g}_{1}) + \mathcal{L}(\boldsymbol{g}_{2}). \end{cases}$$
(4.224)

Combining this, item (III), and item (V) with [3, Proposition 2.19] and the fact that for all  $x \in [2, \infty)$  it holds that  $\lceil \log_2(x) \rceil \leq \log_2(x) + 1 \leq x$  ensures that

$$\begin{aligned} \mathcal{P}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) \\ &= \sum_{k=1}^{\mathcal{L}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1})} \mathbb{D}_{k}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1})(\mathbb{D}_{k-1}(g_{2} \bullet \mathbb{I}_{1} \bullet g_{1}) + 1) \\ &= \left[\sum_{k=1}^{\mathcal{L}(g_{1})-1} \mathbb{D}_{k}(g_{1})(\mathbb{D}_{k-1}(g_{1}) + 1)\right] + 2(\mathbb{D}_{\mathcal{L}(g_{1})-1}(g_{1}) + 1) + \left[\sum_{k=1}^{\mathcal{L}(g_{2})-1} 2(2+1)\right] + 1(2+1) \\ &= \left[\sum_{k=1}^{\mathcal{L}(g_{1})} \mathbb{D}_{k}(g_{1})(\mathbb{D}_{k-1}(g_{1}) + 1)\right] + \mathbb{D}_{\mathcal{L}(g_{1})-1}(g_{1}) + 1 + 6(\mathcal{L}(g_{2}) - 1) + 3 \\ &= \mathcal{P}(g_{1}) + 6(\mathcal{L}(g_{2}) - 1) + 6 \\ &\leq 2304R\kappa^{2}d\varepsilon^{-2} + 6n + 6 \\ &= 2304R\kappa^{2}d\varepsilon^{-2} + 6\max\{\lceil \log_{2}(\kappa) \rceil, 1\} + 6 \\ &\leq 2310R\kappa^{2}d\varepsilon^{-2} + 6\max\{\kappa, 1\} \\ &\leq 2316R\max\{\kappa^{3}, 1\}d\varepsilon^{-2}. \end{aligned}$$

This, (4.220), (4.221), (4.222), (4.223), and (4.224) establish items (i), (ii), (iii), (iv), (v), and (vi). The proof of Corollary 4.30 is thus complete.

**Corollary 4.31.** Let  $a \in \mathbb{R}$ ,  $b \in [a, \infty)$ ,  $d \in \mathbb{N}$   $\mathfrak{c}, \gamma, \kappa \in (0, \infty)$ ,  $\varepsilon \in (0, \kappa)$  satisfy  $\mathfrak{c} \geq 4634 \max\{\kappa^3, 1\} \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil$ , and let  $g \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x, y \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  that  $|g(0)| \leq 2\kappa$ ,  $g(x + 2k\pi) = g(x)$ , and  $|g(x) - g(y)| \leq \kappa |x - y|$ . Then there exists  $\mathfrak{f} \in \mathbb{N}$  such that

- (i) it holds that  $\mathcal{R}(\mathbf{f}) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (*ii*) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left| g\left(\gamma 2^d \sum_{i=1}^d x_i\right) (\mathcal{R}(\boldsymbol{\ell}))(x) \right| \leq \varepsilon$ ,
- (iii) it holds that  $\mathcal{L}(\not l) \leq \mathfrak{c}d$ ,
- (iv) it holds that  $\mathcal{P}(\mathbf{f}) \leq \mathfrak{c} d^2 \varepsilon^{-2}$ , and
- (v) it holds that  $\mathcal{S}(\mathbf{p}) \leq 1$
- (cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Corollary 4.31. Throughout this proof let  $c = 2316 \lceil \log_2(\max\{1, \gamma\} \max\{2, |a|, |b|\}) \rceil \in \mathbb{N}$ . Note that Corollary 4.30 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, \kappa \curvearrowleft \kappa, \varepsilon \curvearrowleft \varepsilon$  in the notation of Corollary 4.30) shows that there exists  $g \in \mathbb{N}$  which satisfies that

- (I) it holds that  $\mathcal{R}(q) \in C(\mathbb{R}^d, \mathbb{R})$ ,
- (II) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left| g\left(\gamma 2^d \sum_{i=1}^d x_i\right) (\mathcal{R}(g))(x) \right| \leq \varepsilon$ ,
- (III) it holds that  $\mathbb{D}_{\mathcal{H}(q)}(q) = 2$ ,

(IV) it holds that  $\mathcal{L}(q) \leq 20^{-1} \max\{\kappa, 1\} cd$ ,

- (V) it holds that  $\mathcal{P}(g) \leq \max\{\kappa^3, 1\} c d\varepsilon^{-2}$ , and
- (VI) it holds that  $\mathcal{S}(q) \leq 2$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8). This, the fact that  $\max\{\kappa, 1\} \leq \max\{\kappa^3, 1\} \min\{\varepsilon^{-2}, 1\}$ , and Lemma 4.3 (applied with  $\not \leftarrow \not q$ ,  $d \leftarrow d$  in the notation of Lemma 4.3) show that there exists  $\not \in \mathbf{N}$  which satisfies that

- (A) it holds that  $\mathcal{R}(\not) \in C(\mathbb{R}^d, \mathbb{R}),$
- (B) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left| g\left(\gamma 2^d \sum_{i=1}^d x_i\right) (\mathcal{R}(\mathbf{f}))(x) \right| \leq \varepsilon$ ,
- (C) it holds that  $\mathcal{L}(\mathbf{\ell}) \leq 2(20^{-1}\max\{\kappa, 1\}cd) + 1 \leq \mathfrak{c}d$ ,
- (D) it holds that

$$\mathcal{P}(\boldsymbol{\ell}) \leq \max\{\kappa^3, 1\} c d\varepsilon^{-2} + 2 + \max\{\kappa, 1\} c d \leq \max\{\kappa^3, 1\} (2c+2) d\varepsilon^{-2} \leq \mathfrak{c} d\varepsilon^{-2}, \quad (4.226)$$

and

(E) it holds that  $\mathcal{S}(\not e) \leq 1$ .

Observe that items (A), (B), (C), (D), and (E) establish items (i), (ii), (iii), (iv), and (v). The proof of Corollary 4.31 is thus complete.  $\Box$ 

# 5 Lower and upper bounds for the minimal number of ANN parameters in the approximation of certain highdimensional functions

In this section we establish in Theorem 5.1, Theorem 5.2, Theorem 5.9, and Corollary 5.10 below that certain families of functions can be approximated without the curse of dimensionality by deep ANNs but neither by shallow nor insufficiently deep ANNs.

Specifically, Theorem 5.1 proves that the plane vanilla *product functions* can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs if the absolute values of the ANN parameters are polynomially bounded in the input dimension but can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1. Our proof of Theorem 5.1 employs

- the lower bound result for the minimal number of parameters of ANNs to approximate the product functions in Corollary 3.4 and
- the upper bound result for the minimal number of parameters of ANNs to approximate the product functions in Corollary 4.16.

Note that Theorem 1.4 in the introduction is a direct consequence of Theorem 5.1.

Theorem 5.2 proves that compositions of certain periodic functions and certain scaled product functions can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs even if the ANN parameters may be arbitrarily large but can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1. Our proof of Theorem 5.2 employs

- the lower bound result for the minimal number of parameters of ANNs to approximate the considered compositions in Proposition 3.21 and
- the upper bound result for the minimal number of parameters of ANNs to approximate the considered compositions in Corollary 4.28.

Observe that Theorem 1.3 in the introduction follows immediatly from Theorem 5.2.

Theorem 5.9 proves that compositions of certain periodic functions and certain scaled sum functions can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs even if the ANN parameters may be arbitrarily large but can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1. Our proof of Theorem 5.9 employs

- the lower bound result for the minimal number of parameters of ANNs to approximate the considered compositions in Proposition 3.22 and
- the upper bound result for the minimal number of parameters of ANNs to approximate the considered compositions in Corollary 4.31.

Theorem 5.9 and the elementary result regarding multidimensional localizing functions in Corollary 5.8 imply Corollary 5.10. Corollary 5.10 establishes the existence of *smooth and uniformly globally bounded functions with compact support* which can neither be approximated without the curse of dimensionality by means of shallow ANNs nor insufficiently deep ANNs even if the ANN parameters may be arbitrarily large but which can be approximated without the curse of dimensionality by sufficiently deep ANNs even if the absolute values of the ANN parameters are assumed to be uniformly bounded by 1. Note that Theorem 1.2 in the introduction is a direct consequence of Corollary 5.10.

# 5.1 ANN approximations regarding high-dimensional product functions

**Theorem 5.1.** Let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$  satisfy  $\max\{|a|, |b|\} \geq 2$  and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \prod_{i=1}^d x_i$ . Then

(i) it holds for all  $c \in [1, \infty)$ ,  $d, L \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land \\ (\mathcal{L}(\not l) \leq L) \land (\mathcal{S}(\not l) \leq cd^c) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq (4cL)^{-3c} 2^{\frac{d}{2L}}$$
(5.1)

and

(ii) it holds for all  $c \in [2^{16} \ln(\max\{|a|, |b|\}), \infty)$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1/2)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not f \in \mathbf{N} : (\mathcal{P}(\not f) = p) \land \\ (\mathcal{L}(\not f) \leq cd^2 |\ln(\varepsilon)|) \land \\ (\mathcal{S}(\not f) \leq 1) \land (\mathcal{R}(\not f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not f))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq cd^3 |\ln(\varepsilon)| \quad (5.2)$$

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Theorem 5.1. Observe that Corollary 3.4 (applied with  $a \curvearrowleft a, b \curvearrowleft b, c \curvearrowleft c, \varepsilon \curvearrowleft \varepsilon, d \backsim d, L \curvearrowleft L, f \curvearrowleft f_d$  for  $c \in [1, \infty), d, L \in \mathbb{N}, \varepsilon \in (0, 1]$  in the notation of Corollary 3.4) demonstrates that for all  $c \in [1, \infty), d, L \in \mathbb{N}, \varepsilon \in (0, 1]$  it holds that

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not l \in \mathbf{N} \colon (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \le L) \land \\ (\mathcal{S}(\not l) \le cd^c) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge (4cL)^{-3c} 2^{\frac{d}{2L}}$$
(5.3)

(cf. Definitions 2.1, 2.3, and 2.13). Hence we obtain item (i). Note that Corollary 4.16 (applied with  $d \curvearrowleft d$ ,  $\varepsilon \curvearrowleft \varepsilon$ ,  $a \curvearrowleft a$ ,  $b \backsim b$ ,  $\gamma \curvearrowleft 1$ ,  $\beta \curvearrowleft 1$  for  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \frac{1}{2})$  in the notation of Corollary 4.16) shows that for all  $c \in [\ln(2)^{-2}12143 \ln(\max\{|a|, |b|\}), \infty)$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \frac{1}{2})$  there exists  $\not{e} \in \mathbb{N}$  such that

(I) it holds that  $\mathcal{R}(\mathscr{L}) \in C(\mathbb{R}^d, \mathbb{R})$ ,

(II) it holds that  $\sup_{x=(x_1,\dots,x_d)\in[a,b]^d} \left|\prod_{i=1}^d x_i - (\mathcal{R}(\mathcal{I}))(x)\right| \le \varepsilon$ ,

- (III) it holds that  $\mathcal{L}(\not) \leq cd^2 |\ln(\varepsilon)|,$
- (IV) it holds that  $\mathcal{P}(\boldsymbol{\ell}) \leq cd^3 |\ln(\varepsilon)|$ , and
- (V) it holds that  $\mathcal{S}(\not l) \leq 1$

Observe that items (I), (II), (III), (IV), and (V) establish item (ii). The proof of Theorem 5.1 is thus complete.  $\hfill \Box$ 

**Theorem 5.2.** Let  $\varphi \in \mathbb{R}$ ,  $\gamma \in (0,1]$ ,  $\beta \in [1,\infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a + 2\pi\beta^{-1},\infty)$ ,  $\mathfrak{c}, \kappa \in (0,\infty)$ satisfy  $\mathfrak{c} \geq 13968\lceil \log_2(\max\{2, |a|, |b|, \beta\}) \rceil \max\{1, \kappa^3\}$  and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \kappa \sin(\gamma \beta^d (\prod_{i=1}^d x_i) + \varphi)$ . Then

(i) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{\max\{1,L-1\}}}$$
(5.4)

and

(ii) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d^{2}\varepsilon^{-1}) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^{d}, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^{d}} |(\mathcal{R}(\not l))(x) - f_{d}(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^{3}\varepsilon^{-2}$$
(5.5)

(cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Theorem 5.2. Throughout this proof let  $g: \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that  $g(x) = \kappa \sin(x+\varphi)$ . Note that Corollary 4.28 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, \kappa \curvearrowleft \kappa, \mathfrak{c} \curvearrowleft \mathfrak{c}, \varepsilon \curvearrowleft \mathfrak{c}, \gamma \backsim \gamma, g \backsim g, f \backsim f_d$  for  $d \in \mathbb{N}, \varepsilon \in (0, \kappa)$  in the notation of Corollary 4.28) implies that for all  $d \in \mathbb{N}, \varepsilon \in (0, \kappa)$  it holds that

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not f \in \mathbf{N} \colon (\mathcal{P}(\not f) = p) \land (\mathcal{L}(\not f) \leq \mathfrak{c}d^{2}\varepsilon^{-1}) \land \\ (\mathcal{S}(\not f) \leq 1) \land (\mathcal{R}(\not f) \in C(\mathbb{R}^{d}, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^{d}} |(\mathcal{R}(\not f))(x) - f_{d}(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^{3}\varepsilon^{-2}.$$
 (5.6)

Furthermore, observe that Proposition 3.21 (applied with  $\varphi \frown \varphi, \gamma \frown \gamma, \beta \frown \beta, a \frown a, b \frown b, \\ \kappa \frown \kappa, f_d \frown f_d \text{ for } d \in \mathbb{N} \text{ in the notation of Proposition 3.21}) shows that for all <math>d, L \in \mathbb{N}, \\ \varepsilon \in (0, \kappa) \text{ it holds that}$ 

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not l \in \mathbf{N} \colon (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{\max\{1,L-1\}}}.$$
 (5.7)

This and (5.6) establish items (i) and (ii). The proof of Theorem 5.2 is thus complete.

**Corollary 5.3.** Let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $\kappa \in (0, \infty)$  and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \kappa \sin\left(\left(\frac{2\pi}{b-a}\right)^d\left(\prod_{i=1}^d x_i\right)\right)$ . Then there exists  $\mathfrak{c} \in \mathbb{R}$  such that

(i) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not e \in \mathbf{N} : (\mathcal{P}(\not e) = p) \land (\mathcal{L}(\not e) \leq L) \land \\ (\mathcal{R}(\not e) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not e))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{L}}$$
(5.8)

and

(ii) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d^{2}\varepsilon^{-1}) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^{d}, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^{d}} |(\mathcal{R}(\not l))(x) - f_{d}(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^{3}\varepsilon^{-2}$$
(5.9)

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Corollary 5.3. Note that Theorem 5.2 (applied with  $\varphi \curvearrowleft 0, \gamma \curvearrowleft 1, \beta \curvearrowleft \frac{2\pi}{b-a}, a \backsim a, b \backsim b, \kappa \curvearrowleft \kappa, f_d \curvearrowleft f_d$  for  $d \in \mathbb{N}$  in the notation of Theorem 5.2) shows items (i) and (ii). The proof of Corollary 5.3 is thus complete.

#### 5.2 Localizing functions

**Lemma 5.4.** Let  $f : \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that

$$f(x) = \begin{cases} 0 & : x \le 0\\ e^{-\frac{1}{x}} & : x > 0. \end{cases}$$
(5.10)

Then

- (i) it holds that  $f \in C^{\infty}(\mathbb{R}, \mathbb{R})$ .
- (ii) it holds for all  $x \in \mathbb{R}$  that  $|f'(x)| \leq 1$  and

Proof of Lemma 5.4. Observe that (5.10) ensures that

$$\lim_{h \searrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \searrow 0} \frac{1}{he^{\frac{1}{h}}} = 0 = \lim_{h \nearrow 0} \frac{f(h) - f(0)}{h}.$$
(5.11)

Combining this with (5.10) demonstrates that for all  $x \in \mathbb{R}$  it holds that

$$f \in C^1(\mathbb{R}, [0, 1])$$
 and  $f'(x) = \begin{cases} 0 & : x \le 0\\ \frac{1}{x^2}e^{-\frac{1}{x}} & : x > 0. \end{cases}$  (5.12)

This ensures that

$$\lim_{h \searrow 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \searrow 0} \frac{1}{h^3 e^{\frac{1}{h}}} = 0 = \lim_{h \nearrow 0} \frac{f'(h) - f'(0)}{h}.$$
(5.13)

Furthermore, note that the chain and the product rule ensure that for all  $g \in C^1(\mathbb{R}, [0, 1])$ ,  $x \in (0, \infty)$  with  $\forall y \in (0, \infty) \colon g(y) = \frac{1}{y^2} e^{-\frac{1}{y}}$  it holds that

$$g'(x) = -\frac{2}{x^3}e^{-\frac{1}{x}} + \frac{1}{x^4}e^{-\frac{1}{x}} = \left(\frac{1}{x} - 2\right)\frac{1}{x^3}e^{-\frac{1}{x}}.$$
(5.14)

Combining this, (5.12), and (5.13) shows that

$$\sup_{x \in \mathbb{R}} |f'(x)| = \left| f'\left(\frac{1}{2}\right) \right| = 4e^{-2} \le 1.$$
(5.15)

This establishes item (ii). Moreover, observe that for all  $n \in \mathbb{N}$  with

$$\exists p \in \mathbb{Z}[X] \,\forall x \in \mathbb{R} \colon f^{(n)}(x) = \begin{cases} 0 & : x \le 0\\ p(\frac{1}{x})e^{-\frac{1}{x}} & : x > 0 \end{cases}$$
(5.16)

it holds that

$$\lim_{h \searrow 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \lim_{h \searrow 0} \frac{p(\frac{1}{h})}{he^{\frac{1}{h}}} = 0 = \lim_{h \nearrow 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h}.$$
 (5.17)

In addition, note that for all  $p \in \mathbb{Z}[X]$ ,  $g \in C^1(\mathbb{R}, [0, 1])$ ,  $x \in (0, \infty)$  with  $\forall y \in (0, \infty)$ :  $g(y) = p(\frac{1}{y})e^{-\frac{1}{y}}$  it holds that

$$g'(x) = -\frac{1}{x^2}p'\left(\frac{1}{x}\right)e^{-\frac{1}{x}} + p\left(\frac{1}{x}\right)\frac{1}{x^2}e^{-\frac{1}{x}} = \left(p\left(\frac{1}{x}\right)\frac{1}{x^2} - \frac{1}{x^2}p'\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x}}.$$
(5.18)

Combining this, (5.10), (5.12), and (5.17) with the fact that for all  $p \in \mathbb{Z}[X]$  it holds that  $p' \in \mathbb{Z}[X]$  and induction ensures that for all  $n \in \mathbb{N}_0$  there exists  $p \in \mathbb{Z}[X]$  such that for all  $x \in \mathbb{R}$  it holds that

$$f^{(n)} \in C^1(\mathbb{R}, \mathbb{R})$$
 and  $f^{(n)}(x) = \begin{cases} 0 & : x \le 0\\ p(\frac{1}{x})e^{-\frac{1}{x}} & : x > 0. \end{cases}$  (5.19)

This establishes item (i). The proof of Lemma 5.4 is thus complete.

**Lemma 5.5.** Let  $\delta \in (0, \infty)$ . Then there exists  $\varphi \in C^{\infty}(\mathbb{R}, [0, 1])$  such that

- (i) it holds for all  $x \in (-\infty, 0]$  that  $\varphi(x) = 0$ ,
- (ii) it holds for all  $x \in (0, \delta)$  that  $\varphi(x) \in (0, 1)$ ,
- (iii) it holds for all  $x \in [\delta, \infty)$  that  $\varphi(x) = 1$ , and
- (iv) it holds for all  $x \in \mathbb{R}$  that  $|\varphi'(x)| \leq \frac{48}{\delta}$ .

Proof of Lemma 5.4. Throughout this proof let  $f \colon \mathbb{R} \to \mathbb{R}$  and  $\varphi \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that

$$f(x) = \begin{cases} 0 & : x \le 0\\ e^{-\frac{\delta}{x}} & : x > 0 \end{cases} \quad \text{and} \quad \varphi(x) = \frac{f(x)}{f(x) + f(\delta - x)}. \tag{5.20}$$

Observe that (5.20), Lemma 5.4, and the fact that for all  $x \in \mathbb{R}$  it holds that  $f(x) + f(\delta - x) \ge f\left(\frac{\delta}{2}\right) = e^{-2} \ge \left(\frac{4}{11}\right)^2 = \frac{16}{121} \ge \frac{1}{8}$  show that for all  $x \in (0, \delta)$  it holds that

$$2f(\delta) \ge f(x) + f(\delta - x) \ge \frac{1}{8}$$
 and  $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R}).$  (5.21)

Furthermore, note that Lemma 5.4, (5.20), and the chain rule demonstrate that for all  $x \in \mathbb{R}$  it holds that

$$|f'(x)| \le \frac{1}{\delta}.\tag{5.22}$$

Moreover, observe that (5.20) ensures that for all  $x \in (-\infty, 0]$  it holds that

$$\varphi(x) = \frac{f(x)}{f(x) + f(\delta - x)} = \frac{0}{0 + e^{-\frac{\delta}{\delta - x}}} = 0.$$
 (5.23)

In addition, note that (5.20) shows that for all  $x \in (0, \delta)$  it holds that

$$0 = \frac{0}{e^{-\frac{\delta}{x}} + e^{-\frac{\delta}{-x}}} < \frac{e^{-\frac{\delta}{x}}}{e^{-\frac{\delta}{x}} + e^{-\frac{\delta}{\delta-x}}} = \varphi(x) = \frac{e^{-\frac{\delta}{x}}}{e^{-\frac{\delta}{x}} + e^{-\frac{\delta}{\delta-x}}} < \frac{e^{-\frac{\delta}{x}}}{e^{-\frac{\delta}{x}}} = 1.$$
 (5.24)

Furthermore, observe that (5.20) demonstrates that for all  $x \in [\delta, \infty)$  it holds that

$$\varphi(x) = \frac{f(x)}{f(x) + f(\delta - x)} = \frac{e^{-\frac{\delta}{x}}}{e^{-\frac{\delta}{x}} + 0} = 1.$$
(5.25)

Moreover, note that (5.20), (5.21), (5.22), Lemma 5.4, the quotient rule, and the fact that  $e^{-1} \leq \frac{10}{27}$  ensure that for all  $x \in (0, \delta)$  it holds that

$$\begin{aligned} |\varphi'(x)| &= \left| \frac{f'(x)(f(x) + f(\delta - x)) - (f'(x) - f'(\delta - x))f(x)}{(f(x) + f(\delta - x))^2} \right| \\ &= \left| \frac{f'(x)f(\delta - x) + f'(\delta - x)f(x)}{(f(x) + f(\delta - x))^2} \right| \\ &\leq \frac{|f'(x)f(\delta - x)| + |f'(\delta - x)f(x)|}{(f(x) + f(\delta - x))^2} \\ &\leq \frac{8^2}{\delta} (|f(\delta - x)| + |f(x)|) \leq \frac{64(2f(\delta))}{\delta} = \frac{128e^{-1}}{\delta} \leq \frac{1280}{27\delta} \leq \frac{48}{\delta}. \end{aligned}$$
(5.26)

This, (5.21), (5.23), (5.24), and (5.25) establish items (i), (ii), (iii), and (iv). The proof of Lemma 5.5 is thus complete.

**Lemma 5.6.** Let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $\delta \in (0, \infty)$ . Then there exists  $\varphi \in C^{\infty}(\mathbb{R}, [0, 1])$  such that

- (i) it holds for all  $x \in (-\infty, a \delta] \cup [b + \delta, \infty)$  that  $\varphi(x) = 0$ ,
- (ii) it holds for all  $x \in (a \delta, a) \cup (b, b + \delta)$  that  $\varphi(x) \in (0, 1)$ ,

- (iii) it holds for all  $x \in [a, b]$  that  $\varphi(x) = 1$ , and
- (iv) it holds for all  $x \in \mathbb{R}$  that  $|\varphi'(x)| \leq \frac{48}{\delta}$ .

Proof of Lemma 5.6. Observe that Lemma 5.5 shows that there exists  $f \in C^{\infty}(\mathbb{R}, [0, 1])$  which satisfies that

- (I) it holds for all  $x \in (-\infty, 0]$  that f(x) = 0,
- (II) it holds for all  $x \in (0, \delta)$  that  $f(x) \in (0, 1)$ ,
- (III) it holds for all  $x \in [\delta, \infty)$  that f(x) = 1, and
- (IV) it holds for all  $x \in \mathbb{R}$  that  $|f'(x)| \leq \frac{48}{\delta}$ .

Next let  $\varphi \colon \mathbb{R} \to [0,1]$  satisfy for all  $x \in \mathbb{R}$  that

$$\varphi(x) = \begin{cases} f(x-a+\delta) & : x < a\\ f(b-x+\delta) & : x \ge a \end{cases}.$$
(5.27)

Note that (5.27) and item (I) demonstrate that for all  $x \in (-\infty, a - \delta]$ ,  $y \in [b + \delta, \infty)$  it holds that

$$\varphi(x) = f(x - a + \delta) = 0$$
 and  $\varphi(y) = f(b - x + \delta) = 0.$  (5.28)

Furthermore, observe that (5.27) and item (II) show that for all  $x \in (a - \delta, a)$ ,  $y \in (b, b + \delta)$  it holds that

$$\varphi(x) = f(x - a + \delta) \in (0, 1)$$
 and  $\varphi(y) = f(b - x + \delta) \in (0, 1).$  (5.29)

Moreover, note that (5.27) and item (III) imply that for all  $x \in [a, b]$  it holds that

$$\varphi(x) = f(b - x + \delta) = 1. \tag{5.30}$$

In addition, observe that (5.27), item (III), and the fact that for all  $k \in \mathbb{N}_0$  it holds that  $f \in C^{\infty}(\mathbb{R}, [0, 1])$  and  $(-1)^k f^{(k)}(b - a + \delta) = f^{(k)}(\delta)$  show that for all  $k \in \mathbb{N}_0$  with  $\varphi \in C^k(\mathbb{R}, [0, 1])$ ,  $\forall x \in (-\infty, a) : \varphi^{(k)}(x) = f^{(k)}(x - a + \delta)$ , and  $\forall x \in [a, \infty) : \varphi^{(k)}(x) = (-1)^k f^{(k)}(b - x + \delta)$  it holds that

$$\lim_{h \neq 0} \frac{\varphi^{(k)}(a+h) - \varphi^{(k)}(a)}{h} = \lim_{h \neq 0} \frac{f^{(k)}(h+\delta) - (-1)^k f^{(k)}(b-a+\delta)}{h} \\
= \lim_{h \neq 0} \frac{f^{(k)}(\delta+h) - f^{(k)}(\delta)}{h} \\
= f^{(k+1)}(\delta) \\
= (-1)^{k+1} f^{(k+1)}(b-a+\delta) \\
= \lim_{h \searrow 0} \frac{\varphi^{(k)}(a+h) - \varphi^{(k)}(a)}{h}.$$
(5.31)

Combining this, (5.27), and the fact that  $\lim_{x \nearrow a} \varphi(x) = 1 = \lim_{x \searrow a} \varphi(x)$  with induction ensures that for all  $x \in (-\infty, a)$ ,  $y \in [a, \infty)$  it holds that

$$\varphi \in C^{\infty}(\mathbb{R}, [0, 1]), \qquad \varphi'(x) = f'(x - a + \delta) \qquad \text{and} \qquad \varphi'(y) = -f'(b - y + \delta).$$
(5.32)

Hence item (IV) demonstrates that for all  $x \in \mathbb{R}$  it holds that

 $|\varphi'(x)| \le \max\{|f'(x-a+\delta)|, |f'(b-x+\delta)|\} \le \frac{48}{\delta}.$ (5.33)

This, (5.28), (5.29), (5.30), and (5.32) establish items (i), (ii), (iii), and (iv). The proof of Lemma 5.6 is thus complete.

**Lemma 5.7.** Let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $\delta \in (0, \infty)$ ,  $d \in \mathbb{N}$ . Then there exists  $\varphi \in C^{\infty}(\mathbb{R}^d, [0, 1])$  such that

- (i) it holds for all  $x \in \mathbb{R}^d \setminus (a \delta, b + \delta)^d$  that  $\varphi(x) = 0$ ,
- (ii) it holds for all  $x \in [a, b]^d$  that  $\varphi(x) = 1$ , and
- (iii) it holds for all  $x, y \in \mathbb{R}^d$  that  $|\varphi(x) \varphi(y)| \le \frac{48d}{\delta} ||x y||_2$

(cf. Definition 3.14).

Proof of Lemma 5.7. Note that Lemma 5.6 (applied with  $a \curvearrowleft a, b \curvearrowleft b, \delta \curvearrowleft \delta$  in the notation of Lemma 5.6) shows that there exists  $f \in C^{\infty}(\mathbb{R}, [0, 1])$  which satisfies that

- (I) it holds for all  $x \in (-\infty, a \delta] \cup [b + \delta, \infty)$  that f(x) = 0,
- (II) it holds for all  $x \in (a \delta, a) \cup (b, b + \delta)$  that  $f(x) \in (0, 1)$ ,
- (III) it holds for all  $x \in [a, b]$  that f(x) = 1, and
- (IV) it holds for all  $x \in \mathbb{R}$  that  $|f'(x)| \leq \frac{48}{\delta}$ .

Next let  $\varphi \in C^{\infty}(\mathbb{R}^d, [0, 1])$  satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$\varphi(x) = \prod_{i=1}^{d} f(x_i). \tag{5.34}$$

Observe that (5.34) and item (I) demonstrate that for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus (a - \delta, b + \delta)^d$  it holds that

$$\varphi(x) = \prod_{i=1}^{d} f(x_i) = 0.$$
(5.35)

Furthermore, note that (5.34) and item (III) demonstrate that for all  $x = (x_1, \ldots, x_d) \in [a, b]^d$  it holds that

$$\varphi(x) = \prod_{i=1}^{d} f(x_i) = \prod_{i=1}^{d} 1 = 1.$$
(5.36)

Moreover, observe that (5.34), item (IV), and the fact that  $f \in C^{\infty}(\mathbb{R}, [0, 1])$  imply that for all  $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$  it holds that

$$\begin{aligned} |\varphi(x) - \varphi(y)| &= \left| \left[ \prod_{i=1}^{d} f(x_{i}) \right] - \left[ \prod_{i=1}^{d} f(y_{i}) \right] \right| \\ &= \left| \sum_{j=1}^{d} \left( \left[ \prod_{i=1}^{j-1} f(y_{i}) \right] \left[ \prod_{i=j}^{d} f(x_{i}) \right] - \left[ \prod_{i=1}^{j} f(y_{i}) \right] \left[ \prod_{i=j+1}^{d} f(x_{i}) \right] \right) \right| \\ &= \left| \sum_{j=1}^{d} (f(x_{j}) - f(y_{j})) \left[ \prod_{i=1}^{j-1} f(y_{i}) \right] \left[ \prod_{i=j+1}^{d} f(x_{i}) \right] \right| \\ &\leq \sum_{j=1}^{d} |f(x_{j}) - f(y_{j})| \left[ \prod_{i=1}^{j-1} |f(y_{i})| \right] \left[ \prod_{i=j+1}^{d} |f(x_{i})| \right] \\ &\leq \sum_{j=1}^{d} |f(x_{j}) - f(y_{j})| \leq \sum_{j=1}^{d} \frac{48}{\delta} |x_{j} - y_{j}| \leq \frac{48d}{\delta} ||x - y||_{2}. \end{aligned}$$
 (5.37)

Combining this, (5.35), and (5.36) establishes items (i), (ii), and (iii). The proof of Lemma 5.7 is thus complete.

**Corollary 5.8.** Let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $d \in \mathbb{N}$ ,  $\kappa, \delta, L \in (0, \infty)$  and  $g \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$  satisfy for all  $x, y \in [a - \delta, b + \delta]^d$  that  $|g(x) - g(y)| \leq L ||x - y||_2$  and  $|g(x)| \leq \kappa$ . Then there exists  $f \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$  such that

- (i) it holds for all  $x \in [a, b]^d$  that f(x) = g(x),
- (ii) it holds for all  $x \in \mathbb{R}^d$  that  $|f(x)| \leq \kappa (\mathbb{1}_{[a-\delta,b+\delta]^d}(x))$ , and
- (iii) it holds for all  $x, y \in \mathbb{R}^d$  that  $|f(x) f(y)| \le \left(\frac{48\kappa d}{\delta} + L\right) ||x y||_2$
- (cf. Definition 3.14).

Proof of Corollary 5.8. Note that Lemma 5.7 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \backsim d, \delta \backsim \delta$  in the notation of Lemma 5.7) shows that there exists  $\varphi \in C^{\infty}(\mathbb{R}^d, [0, 1])$  which satisfies that

- (I) it holds for all  $x \in \mathbb{R}^d \setminus (a \delta, b + \delta)^d$  that  $\varphi(x) = 0$ ,
- (II) it holds for all  $x \in [a, b]^d$  that  $\varphi(x) = 1$ , and
- (III) it holds for all  $x, y \in \mathbb{R}^d$  that  $|\varphi(x) \varphi(y)| \le \frac{48d}{\delta} ||x y||_2$ .

Next let  $f \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$  satisfy for all  $x \in \mathbb{R}^d$  that

$$f(x) = \varphi(x)g(x). \tag{5.38}$$

Observe that (5.38) and item (I) ensure that for all  $x \in \mathbb{R}^d \setminus (a - \delta, b + \delta)^d$  it holds that

$$f(x) = \varphi(x)g(x) = 0. \tag{5.39}$$

Furthermore, note that (5.38) and item (II) ensure that for all  $x \in [a, b]^d$  it holds that

$$f(x) = \varphi(x)g(x) = g(x). \tag{5.40}$$

Moreover, observe that (5.38), item (III), and the fact that  $\varphi \in C^{\infty}(\mathbb{R}^d, [0, 1])$  imply that for all  $x, y \in \mathbb{R}^d$  it holds that  $|f(x)| \leq \kappa$  and

$$|f(x) - f(y)| = |\varphi(x)g(x) - \varphi(y)g(y)|$$

$$\leq |\varphi(x)g(x) - \varphi(y)g(x)| + |\varphi(y)g(x) - \varphi(y)g(y)|$$

$$\leq |\varphi(x) - \varphi(y)||g(x)| + |\varphi(y)||g(x) - g(y)|$$

$$\leq \frac{48\kappa d}{\delta} ||x - y||_2 + L||x - y||_2 \leq (\frac{48\kappa d}{\delta} + L)||x - y||_2.$$
(5.41)

Combining this, (5.38), (5.39), and (5.40) establishes items (i), (ii), and (iii). The proof of Corollary 5.8 is thus complete.

# 5.3 ANN approximations for classes of smooth and bounded functions

**Theorem 5.9.** Let  $\varphi \in \mathbb{R}$ ,  $\gamma, \kappa \in (0, \infty)$ ,  $a \in \mathbb{R}$ ,  $b \in [a + \pi \gamma^{-1}, \infty)$ ,  $\mathfrak{c} \in (0, \infty)$  satisfy  $\mathfrak{c} \geq 4634\lceil \log_2(\max\{1, \gamma\} \max\{|a|, |b|, 2\}) \rceil \max\{\kappa^3, 1\}$  and for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \kappa \sin(\gamma 2^d (\sum_{i=1}^d x_i) + \varphi)$ . Then

(i) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not f \in \mathbf{N} : (\mathcal{P}(\not f) = p) \land (\mathcal{L}(\not f) \leq L) \land \\ (\mathcal{R}(\not f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not f))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{\max\{1,L-1\}}}$$
(5.42)

and

(ii) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^2\varepsilon^{-2} \quad (5.43)$$

(cf. Definitions 2.1, 2.3, 2.13, and 4.8).

Proof of Theorem 5.9. Throughout this proof let  $g: \mathbb{R} \to \mathbb{R}$  satisfy for all  $x \in \mathbb{R}$  that  $g(x) = \kappa \sin(x + \varphi)$ . Note that Corollary 4.31 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, \kappa \curvearrowleft \kappa, \varepsilon \backsim \varepsilon, \gamma \curvearrowleft \gamma, \mathfrak{c} \curvearrowleft \mathfrak{c}, g \curvearrowleft g$  for  $d \in \mathbb{N}, \varepsilon \in (0, \kappa)$  in the notation of Corollary 4.31) implies that for all  $d \in \mathbb{N}, \varepsilon \in (0, \kappa)$  it holds that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^2\varepsilon^{-2}.$$
(5.44)

Furthermore, observe that Proposition 3.22 (applied with  $\varphi \curvearrowleft \varphi, \kappa \curvearrowleft \kappa, \gamma \curvearrowleft \gamma, a \curvearrowleft a, b \curvearrowleft b, f_d \curvearrowleft f_d$  for  $d \in \mathbb{N}$  in the notation of Proposition 3.22) shows that

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not l \in \mathbf{N} \colon (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{\max\{1,L-1\}}}.$$
 (5.45)

This and (5.44) establish items (i) and (ii). The proof of Theorem 5.9 is thus complete.

**Corollary 5.10.** Let  $a \in \mathbb{R}$ ,  $b \in (a, \infty)$ ,  $\kappa, \delta \in (0, \infty)$ . Then there exist  $\mathfrak{c} \in \mathbb{R}$  and  $f_d \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$ ,  $d \in \mathbb{N}$ , with  $\forall d \in \mathbb{N}, x \in \mathbb{R}^d : |f_d(x)| \leq \kappa \mathbb{1}_{[a-\delta,b+\delta]^d}(x)$  such that

(i) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{L}}$$
(5.46)

and

(ii) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^2\varepsilon^{-2} \quad (5.47)$$

(cf. Definitions 2.1, 2.3, and 2.13).

Proof of Corollary 5.10. Throughout this proof let  $g_d \in C^{\infty}(\mathbb{R}^d, \mathbb{R}), d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$g_d(x) = \kappa \sin\left(\frac{2^d \pi}{b-a} \left(\sum_{i=1}^d x_i\right)\right). \tag{5.48}$$

Note that Corollary 5.8 (applied with  $a \curvearrowleft a, b \curvearrowleft b, d \curvearrowleft d, \kappa \curvearrowleft \kappa, \delta \curvearrowleft \delta, g \curvearrowleft g_d$  for  $d \in \mathbb{N}$  in the notation of Corollary 5.8) shows that there exist  $f_d \in C^{\infty}(\mathbb{R}^d, \mathbb{R}), d \in \mathbb{N}$ , which satisfy that

(I) it holds for all 
$$d \in \mathbb{N}$$
,  $x = (x_1, \dots, x_d) \in [a, b]^d$  that  $f_d(x) = \kappa \sin\left(\frac{2^d \pi}{b-a} \left(\sum_{i=1}^d x_i\right)\right)$ , and

(II) it holds for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that  $|f_d(x)| \le \kappa (\mathbb{1}_{[a-\delta,b+\delta]^d}(x)).$ 

Observe that item (I), the fact that for all  $L \in \mathbb{N}$  it hods that  $\max\{L-1,1\} \leq L$  and Theorem 5.9 (applied with  $\varphi \curvearrowleft 0, \gamma \curvearrowleft \frac{\pi}{b-a}, a \curvearrowleft a, b \curvearrowleft b, \kappa \curvearrowleft \kappa, f_d \curvearrowleft f_d$  for  $d \in \mathbb{N}$  in the notation of Theorem 5.9) show that there exists  $\mathfrak{c} \in \mathbb{R}$  such that

(A) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not f \in \mathbf{N} : (\mathcal{P}(\not f) = p) \land (\mathcal{L}(\not f) \le L) \land \\ (\mathcal{R}(\not f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} | (\mathcal{R}(\not f))(x) - f_d(x)| \le \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \ge 2^{\frac{d}{L}}$$
(5.49)

and

(B) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^2\varepsilon^{-2} \quad (5.50)$$

Combining item (I), item (II), item (A), and item (B) establishes item (i) and item (ii). The proof of Corollary 5.10 is thus complete.  $\Box$ 

**Corollary 5.11.** Let  $\kappa \in (0, \infty)$ . Then there exist  $\mathfrak{c} \in (0, \infty)$  and  $f_d \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$ ,  $d \in \mathbb{N}$ , with compact support such that for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  it holds that  $|f_d(x)| \leq \kappa$ ,  $|f_d(x) - f_d(y)| \leq 2\kappa d ||x - y||_2$ , and

(i) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq L) \land \\ (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [-2^d, 2^d]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{\max\{1, L-1\}}}$$
(5.51)

and

(ii) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not l \in \mathbf{N} : (\mathcal{P}(\not l) = p) \land (\mathcal{L}(\not l) \leq \mathfrak{c}d) \land \\ (\mathcal{S}(\not l) \leq 1) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [-2^d, 2^d]^d} | (\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^2\varepsilon^{-2} \quad (5.52)$$

(cf. Definitions 2.1, 2.3, 2.13, and 3.14).

Proof of Corollary 5.11. Throughout this proof let  $g_d \in C^{\infty}(\mathbb{R}^d, \mathbb{R}), d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$g_d(x) = \kappa \sin\left(\sum_{i=1}^d x_i\right). \tag{5.53}$$

Note that (5.53) shows that for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d)$ ,  $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$  it holds that

$$|g_{d}(x) - g_{d}(y)| = |\kappa \sin\left(\sum_{i=1}^{d} x_{i}\right) - \kappa \sin\left(\sum_{i=1}^{d} y_{i}\right)| \\ = \kappa |\sin\left(\sum_{i=1}^{d} x_{i}\right) - \sin\left(\sum_{i=1}^{d} y_{i}\right)| \\ \leq \kappa |\left(\sum_{i=1}^{d} x_{i}\right) - \left(\sum_{i=1}^{d} y_{i}\right)| \\ = \kappa |\left(\sum_{i=1}^{d} x_{i}\right) - \left(\sum_{i=1}^{d} y_{i}\right)| \\ \leq \kappa d ||x - y||_{2}$$
(5.54)

(cf. Definition 3.14). This and Corollary 5.8 (applied with  $a \cap -2^d$ ,  $b \cap 2^d$ ,  $d \cap d$ ,  $\kappa \cap \kappa$ ,  $\delta \cap 48$ ,  $L \cap \kappa d$ ,  $g \cap g_d$  for  $d \in \mathbb{N}$  in the notation of Corollary 5.8) shows that there exist  $f_d \in C^{\infty}(\mathbb{R}^d, \mathbb{R}), d \in \mathbb{N}$ , which satisfy that

- (I) it holds for all  $d \in \mathbb{N}$ ,  $x = (x_1, \dots, x_d) \in [-2^d, 2^d]^d$  that  $f_d(x) = \kappa \sin\left(\left(\sum_{i=1}^d x_i\right) + \varphi\right)$ ,
- (II) it holds for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that  $|f_d(x)| \le \kappa (\mathbb{1}_{[-2^d 48, 2^d + 48]^d}(x))$ , and
- (III) it holds for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  that  $|f_d(x) f_d(y)| \le 2\kappa d ||x y||_2$ .

Observe that item (I), the fact that for all  $L \in \mathbb{N}$  it hods that  $\max\{L-1,1\} \leq L$  and Theorem 5.9 (applied with  $\varphi \curvearrowleft 0$ ,  $\gamma \curvearrowleft 2^{-d}$ ,  $a \backsim -2^d$ ,  $b \backsim 2^d$ ,  $\kappa \backsim \kappa$ ,  $f_d \curvearrowleft f_d$  for  $d \in \mathbb{N}$  in the notation of Theorem 5.9) show that there exists  $\mathfrak{c} \in \mathbb{R}$  such that

(A) it holds for all  $d, L \in \mathbb{N}, \varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} : \begin{bmatrix} \exists \not e \in \mathbf{N} : (\mathcal{P}(\not e) = p) \land (\mathcal{L}(\not e) \leq L) \land \\ (\mathcal{R}(\not e) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [-2^d, 2^d]^d} | (\mathcal{R}(\not e))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \geq 2^{\frac{d}{\max\{1, L-1\}}}$$
(5.55)

and

(B) it holds for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, \kappa)$  that

$$\min\left(\left\{p \in \mathbb{N} \colon \begin{bmatrix} \exists \not f \in \mathbf{N} \colon (\mathcal{P}(\not f) = p) \land (\mathcal{L}(\not f) \leq \mathfrak{c}d) \land \\ (\mathcal{S}(\not f) \leq 1) \land (\mathcal{R}(\not f) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [-2^d, 2^d]^d} | (\mathcal{R}(\not f))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right) \leq \mathfrak{c}d^2\varepsilon^{-2} \quad (5.56)$$

(cf. Definitions 2.1, 2.3, and 2.13). Combining item (II), item (III), item (A), and item (B) establishes items (i) and (ii). The proof of Corollary 5.11 is thus complete.  $\Box$ 

# 5.4 Necessity of depth for ANN aproximations with respect to computational capacities

**Corollary 5.12.** Let  $a \in \mathbb{R}$ ,  $b \in [a + 7, \infty)$ , for every  $d \in \mathbb{N}$  let  $f_d \colon \mathbb{R}^d \to \mathbb{R}$  satisfy for all  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $f_d(x) = \sin(\prod_{i=1}^d x_i)$ , and let  $\operatorname{cost} \colon \mathbb{N} \times [0, \infty]^2 \to \mathbb{R}$  satisfy for all  $d \in \mathbb{N}$ ,  $L, \varepsilon \in [0, \infty]$  that

$$\operatorname{cost}(d, L, \varepsilon) = \inf\left(\left\{c \in \mathbb{R} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\max\{1, \ln(\mathcal{S}(\not l))\}\mathcal{P}(\not l) = c) \land \\ (\mathcal{L}(\not l) \leq L) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f_d(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right). \quad (5.57)$$

Then there exists  $\mathfrak{c} \in (0,\infty)$  such that for all  $d, L \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  it holds that

$$cost(d, L, \varepsilon) \ge 2^{\frac{d}{L}} \quad and \quad cost(d, \mathfrak{c}d^2\varepsilon^{-1}, \varepsilon) \le \mathfrak{c}d^3\varepsilon^{-2}.$$
(5.58)

Proof of Corollary 5.12. Note that Theorem 5.2 (applied with  $\varphi \curvearrowleft 0, \gamma \curvearrowleft 1, \beta \curvearrowleft 1, a \backsim a, b \backsim b, \kappa \backsim 1, f_d \curvearrowleft f_d$  for  $d \in \mathbb{N}$  in the notation of Theorem 5.2) shows (5.58). The proof of Corollary 5.12 is thus complete.

**Corollary 5.13.** Let  $a \in \mathbb{R}$ ,  $b \in [a + 4, \infty)$  and let  $\text{cost}: (\bigcup_{d \in \mathbb{N}} C(\mathbb{R}^d, \mathbb{R})) \times [0, \infty]^2 \to \mathbb{R}$  satisfy for all  $d \in \mathbb{N}$ ,  $f \in C(\mathbb{R}^d, \mathbb{R})$ ,  $L, \varepsilon \in [0, \infty]$  that

$$\operatorname{cost}(f, L, \varepsilon) = \inf\left(\left\{c \in \mathbb{R} : \begin{bmatrix} \exists \not l \in \mathbb{N} : (\max\{1, \ln(\mathcal{S}(\not l))\}\mathcal{P}(\not l) = c) \land \\ (\mathcal{L}(\not l) \leq L) \land (\mathcal{R}(\not l) \in C(\mathbb{R}^d, \mathbb{R})) \land \\ (\sup_{x \in [a,b]^d} |(\mathcal{R}(\not l))(x) - f(x)| \leq \varepsilon) \end{bmatrix}\right\} \cup \{\infty\}\right). \quad (5.59)$$

Then there exist  $\mathbf{c} \in (0, \infty)$  and infinitely often differentiable  $f_d \colon \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}$ , with compact support and  $\sup_{d \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |f_d(x)| \leq 1$  such that for all  $d, L \in \mathbb{N}, \varepsilon \in (0, 1)$  it holds that

$$cost(f_d, L, \varepsilon) \ge 2^{\frac{d}{L}} \quad and \quad cost(f_d, \mathfrak{c}d, \varepsilon) \le \mathfrak{c}d^2\varepsilon^{-2}.$$
(5.60)

Proof of Corollary 5.13. Observe that Theorem 5.9 (applied with  $\varphi \curvearrowleft 0, \gamma \backsim 1, a \curvearrowleft a, b \backsim b, \\ \kappa \backsim 1$  in the notation of Theorem 5.9) shows (5.60). The proof of Corollary 5.13 is thus complete.

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# References

- BECK, C., JENTZEN, A., AND KUCKUCK, B. Full error analysis for the training of deep neural networks. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 25, 2 (2022), Paper No. 2150020, 76.
- [2] BELLMAN, R. Dynamic programming. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2010. Reprint of the 1957 edition, With a new introduction by Stuart Dreyfus.
- [3] BENEVENTANO, P., CHERIDITO, P., GRAEBER, R., JENTZEN, A., AND KUCK-UCK, B. Deep neural network approximation theory for high-dimensional functions. arXiv:2112.14523 (2021).
- [4] CHEN, L., AND WU, C. A note on the expressive power of deep rectified linear unit networks in high-dimensional spaces. *Math. Methods Appl. Sci.* 42, 9 (2019), 3400–3404.
- [5] CHEN, T., LU, S., AND FAN, J. SS-HCNN: semi-supervised hierarchical convolutional neural network for image classification. *IEEE Trans. Image Process.* 28, 5 (2019), 2389– 2398.
- [6] CHERIDITO, P., JENTZEN, A., AND ROSSMANNEK, F. Efficient approximation of highdimensional functions with neural networks. *IEEE Trans. Neural Netw. Learn. Syst. 33*, 7 (2022), 3079–3093.
- [7] CHUI, C. K., LIN, S.-B., AND ZHOU, D.-X. Deep neural networks for rotation-invariance approximation and learning. *Anal. Appl. (Singap.)* 17, 5 (2019), 737–772.
- [8] DANIELY, A. Depth separation for neural networks. In Proceedings of the 2017 Conference on Learning Theory (07–10 Jul 2017), S. Kale and O. Shamir, Eds., vol. 65 of Proceedings of Machine Learning Research, PMLR, pp. 690–696.
- [9] DEVLIN, J., CHANG, M., LEE, K., AND TOUTANOVA, K. BERT: Pre-training of Deep Bidirectional Transformers for Language Understanding. *arXiv:1810.04805* (2018).
- [10] ELBRÄCHTER, D., GROHS, P., JENTZEN, A., AND SCHWAB, C. DNN expression rate analysis of high-dimensional PDEs: application to option pricing. *Constr. Approx.* 55, 1 (2022), 3–71.
- [11] ELDAN, R., AND SHAMIR, O. The power of depth for feedforward neural networks. In Proceedings of the 29th Annual Conference on Learning Theory (Columbia University, New York, New York, USA, 23–26 Jun 2016), V. Feldman, A. Rakhlin, and O. Shamir, Eds., vol. 49 of Proceedings of Machine Learning Research, PMLR, pp. 907–940.
- [12] GROHS, P., HORNUNG, F., JENTZEN, A., AND ZIMMERMANN, P. Space-time error estimates for deep neural network approximations for differential equations. *Adv. Comput. Math.* 49, 4 (2022).

- [13] GROHS, P., IBRAGIMOV, S., JENTZEN, A., AND KOPPENSTEINER, S. Lower bounds for artificial neural network approximations: A proof that shallow neural networks fail to overcome the curse of dimensionality. arXiv:2103.04488 (2021), 53 pages. Accepted in J. Complexity.
- [14] GROHS, P., JENTZEN, A., AND SALIMOVA, D. Deep neural network approximations for solutions of PDEs based on Monte Carlo algorithms. *Partial Differ. Equ. Appl. 3*, 4 (2022), Paper No. 45, 41.
- [15] HE, K., ZHANG, X., REN, S., AND SUN, J. Deep residual learning for image recognition. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR) (June 2016).
- [16] HUANG, G., LIU, Z., VAN DER MAATEN, L., AND WEINBERGER, K. Q. Densely connected convolutional networks. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR) (July 2017).
- [17] JOVANOVIĆ, B. S., AND SÜLI, E. Analysis of finite difference schemes, vol. 46 of Springer Series in Computational Mathematics. Springer, London, 2014. For linear partial differential equations with generalized solutions.
- [18] LIN, T.-Y., MAIRE, M., BELONGIE, S., BOURDEV, L., GIRSHICK, R., HAYS, J., PER-ONA, P., RAMANAN, D., ZITNICK, C. L., AND DOLLÁR, P. Microsoft coco: Common objects in context, 2014.
- [19] MIN, S., LEE, B., AND YOON, S. Deep learning in bioinformatics. Briefings in bioinformatics 18, 5 (2017), 851–869.
- [20] NOVAK, E., AND WOŹNIAKOWSKI, H. Tractability of multivariate problems. Vol. 1: Linear information, vol. 6 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [21] NOVAK, E., AND WOŹNIAKOWSKI, H. Tractability of multivariate problems. Volume II: Standard information for functionals, vol. 12 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2010.
- [22] PALTRINIERI, N., COMFORT, L., AND RENIERS, G. Learning about risk: Machine learning for risk assessment. Safety Science 118 (2019), 475–486.
- [23] PETERSEN, P., AND VOIGTLAENDER, F. Optimal approximation of piecewise smooth functions using deep ReLU neural networks. *Neural Netw.* 108 (2018), 296–330.
- [24] RAGHU, M., POOLE, B., KLEINBERG, J., GANGULI, S., AND SOHL-DICKSTEIN, J. On the expressive power of deep neural networks. In *Proceedings of the International Conference on Machine Learning* (2017), PMLR, pp. 2847–2854.
- [25] RIEGLER, P., AND BIEHL, M. On-line backpropagation in two-layered neural networks. Journal of Physics A: Mathematical and General 28, 20 (oct 1995), L507.

- [26] RUSSAKOVSKY, O., DENG, J., SU, H., KRAUSE, J., SATHEESH, S., MA, S., HUANG, Z., KARPATHY, A., KHOSLA, A., BERNSTEIN, M., BERG, A. C., AND FEI-FEI, L. ImageNet large scale visual recognition challenge. *Int. J. Comput. Vis.* 115, 3 (2015), 211–252.
- [27] SAAD, D., AND SOLLA, S. Dynamics of on-line gradient descent learning for multilayer neural networks. In Advances in Neural Information Processing Systems (1995), D. Touretzky, M. Mozer, and M. Hasselmo, Eds., vol. 8, MIT Press.
- [28] SAFRAN, I., AND SHAMIR, O. Depth-width tradeoffs in approximating natural functions with neural networks. In *Proceedings of the 34th International Conference on Machine Learning* (06–11 Aug 2017), D. Precup and Y. W. Teh, Eds., vol. 70 of *Proceedings of Machine Learning Research*, PMLR, pp. 2979–2987.
- [29] SIDEY-GIBBONS, J. A., AND SIDEY-GIBBONS, C. J. Machine learning in medicine: a practical introduction. BMC medical research methodology 19, 1 (2019), 1–18.
- [30] TADMOR, E. A review of numerical methods for nonlinear partial differential equations. Bull. Amer. Math. Soc. (N.S.) 49, 4 (2012), 507–554.
- [31] TELGARSKY, M. Representation benefits of deep feedforward networks. arXiv:1509.08101 (2015).
- [32] TELGARSKY, M. Benefits of depth in neural networks. arXiv:1602.04485 (2016).
- [33] VENTURI, L., JELASSI, S., OZUCH, T., AND BRUNA, J. Depth separation beyond radial functions. arXiv:2102.01621 (2021).
- [34] WANG, A., SINGH, A., MICHAEL, J., HILL, F., LEVY, O., AND BOWMAN, S. R. Glue: A multi-task benchmark and analysis platform for natural language understanding. arXiv preprint arXiv:1804.07461 (2018).
- [35] WILLIAMS, A., NANGIA, N., AND BOWMAN, S. A Broad-Coverage Challenge Corpus for Sentence Understanding through Inference. In Proceedings of the 2018 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies, Volume 1 (Long Papers) (2018), Association for Computational Linguistics, pp. 1112–1122.
- [36] YU, A., BECQUEY, C., HALIKIAS, D., MALLORY, M. E., AND TOWNSEND, A. Arbitrary-depth universal approximation theorems for operator neural networks. *arXiv:2109.11354* (2021).