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Uncertainty Inequality for Radon Transform on the Heisenberg Group

Jinsen Xiao[1] · **Jianxun He**[2]

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Abstract This article presents the Heisenberg–Pauli–Weyl uncertainty inequality for the Radon transform on the Heisenberg group, which indicates that the Radon transform and the Fourier transform of a nonzero function can not both be sharply localized. The proof is mainly based on some estimates related to the heat kernel, together with the relation between the sublaplacian and the group Fourier transform.

Keywords Heisenberg group · Radon transform · Uncertainty inequality · Heat kernel

Mathematics Subject Classification 43A85 · 44A12 · 52A38

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✉ Jianxun He
hejianxun@gzhu.edu.cn

Jinsen Xiao
jinsenxiao@yahoo.com

1 School of Sciences, Guangdong University of Petrochemical Technology, Maoming 525000, People's Republic of China

2 School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, People's Republic of China

1 Introduction

The classical uncertainty principle in harmonic analysis states that a nonzero function and its Fourier transform cannot both be sharply localized. The precise quantitative formulation of this principle is the Heisenberg inequality:

$$\frac{\|f\|_2^4}{16\pi^2} \leq \int_{\mathbb{R}} |x-a|^2 |f(x)|^2 dx \int_{\mathbb{R}} |\xi - b|^2 |\widehat{f}(\xi)|^2 d\xi.$$

A more general form called Heisenberg–Pauli–Weyl uncertainty inequality on $\mathbb{R}^n$ reads

$$\|f\|_2^4 \leq c_n \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 dx \int_{\mathbb{R}^n} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi, \tag{1.1}$$

which can also be written in the form:

$$\|f\|_2^2 \leq c_n \| |x| f\|_2 \|(-\Delta)^{1/2} f\|_2,$$

where Δ denotes the Laplacian (see [4]).

By the inequality (1.1), analogues inequalities were established by Singer [14] for the wavelet transform and by Wilczok [12] for the Gabor transform (note that those transforms can be treated as the convolution operators). More about the history and the relevance of the uncertainty principle we refer the readers to the survey [4], the books [7,9], and the papers [1,2,10,11].

On the Heisenberg group H^n, Thangavelu [17] proved the following uncertainty inequality:

$$\sqrt{n}\left(\frac{\pi}{2}\right)^{\frac{n+1}{2}} \leq \| |z| f\|_2 \|\mathscr{L}^{1/2} f\|_2,$$

where $\mathscr{L}$ is the Heisenberg sublaplacian. Sitaram et al. [15] obtained a generalized form for $0 \leq \gamma < n+1$,

$$\|f\|_2^2 \leq C \| |(z,t)|^\gamma f\|_2 \|\mathscr{L}^{\gamma/2} f\|_2.$$

We in [20] extended this to a full range for $a, b > 0$,

$$\|f\|_2 \leq C \| |(z,t)|^a f\|_2^{\frac{b}{a+b}} \|\mathscr{L}^{b/2} f\|_2^{\frac{a}{a+b}}. \tag{1.2}$$

We also built a similar inequality for the wavelet transform:

$$\|f\|_2 \leq C \left(\int_0^\infty \int_{H^n} |(z,t)|^{2a} |W_\phi f(z,t,\rho)|^2 \frac{dz dt d\rho}{\rho^{n+2}} \right)^{\frac{b}{a+b}} \|\mathscr{L}^{b/2} f\|_2^{\frac{a}{a+b}},$$

where C depends on ϕ and $a, b > 0$. Obviously, this inequality shows that the wavelet transform and the Fourier transform of a nonzero function can not both be sharply localized.

Now in this paper we aim to extend the Heisenberg–Pauli–Weyl uncertainty inequality for a special singular convolution operator—Heisenberg Radon transform R, which represents an interesting object from the point of view of both harmonic analysis and integral geometry (see [13]). Explicitly, we shall prove the following theorem:

Theorem 1.1 *For* $f \in L^2(H^n)$, $0 \leq a < n+1,\ b \geq 0$, *one has*

$$\|f\|_2 \leq C \||(z,t)|^a Rf\|_2^{\frac{b}{a+b+2n}} \|\mathscr{L}^{b/2} f\|_2^{\frac{a+2n}{a+b+2n}}, \tag{1.3}$$

where C is a constant.

The idea that we modify the inequality (1.2) to inequality (1.3) originates from the following Plancherel formula related to the Radon transform obtained by Geller and Stein [5] and Strichartz [19] respectively:

$$\|(\partial/\partial t)^n Rf\|_2^2 = c_n \|f\|_2^2.$$

The main approach of the proof of Theorem 1.1 is mainly depended on some estimates related to the heat kernel, together with the relation between the sublaplacian and the group Fourier transform.

2 Preliminaries

The Heisenberg group, denoted by H^n, is a nilpotent Lie group of step two whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(z',t') = \left(z + z', t + t' + \frac{1}{2}\mathrm{Im} z\bar{z'}\right).$$

For $(z,t) \in H^n$, the homogeneous norm of (z,t) is given by $|(z,t)| = (|z|^4 + |t|^2)^{1/4}$. Then the ball of radius r centered at (z,t) is defined by $B_r(z,t) = \{(z',t') \in H^n : |(z,t)^{-1}(z',t')| < r\}$. Let S^n be the unit sphere in H^n, for a measurable function f one has (see [3])

$$\int_{H^n} f(z,t)dzdt = \int_{S^n}\int_0^\infty f(r\zeta)r^{2n+1}drd\zeta. \tag{2.1}$$

Let $\pi_\lambda(z,t)$ ($z = x + iy,\ \lambda \in \mathbb{R}\backslash\{0\}$) be the Schrödinger representations acted on $\varphi \in L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)} \varphi(\xi + y).$$

Given a function $f \in L^1(H^n)$, its group Fourier transform $\hat{f}$ is defined to be the operator-valued function and

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

Let $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$, then one has the inversion of Fourier transform

$$f(z,t) = \int_{-\infty}^{\infty} \mathrm{tr}(\pi_\lambda^*(z,t)\hat{f}(\lambda))d\mu(\lambda)$$

and the Plancherel formula

$$\|f\|_2^2 = \int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2 d\mu(\lambda).$$

Suppose f and g are measurable functions on H^n, then their convolution is defined by

$$f * g(z,t) = \int_{H^n} f((z,t)(-w,-s))g(w,s)dwds.$$

It follows from the definition of the Fourier transform that $\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$. In addition, one has the generalized Yong inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

Now consider the Heisenberg sublaplacian

$$\mathscr{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2),$$

where $X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}$. For the Schrödinger representations π_λ one has

$$\pi_\lambda^*(X_j) = i\lambda\xi_j, \quad \pi_\lambda^*(Y_j) = \frac{\partial}{\partial \xi_j}.$$

So that $\pi_\lambda^*(\mathscr{L}) = -\Delta + \lambda^2|\xi|^2 = H(\lambda)$ is the Hermite operator. Let Φ_α ($\alpha \in \mathbb{N}^n$) stand for the normalized Hermite functions on $\mathbb{R}^n$. For $\lambda \in \mathbb{R}^*$, define $\Phi_\alpha^\lambda(\xi) = |\lambda|^{\frac{n}{4}}\Phi_\alpha(|\lambda|^{\frac{1}{2}}\xi)$. Then one has

$$H(\lambda)\Phi_\alpha^\lambda = (2|\alpha| + n)|\lambda|\Phi_\alpha^\lambda.$$

One important relevance of sublaplacian $\mathscr{L}$ is the heat semigroup defined by

$$(e^{-s\mathscr{L}}f)(z,t) = q_s * f(z,t),$$

where q_s is the heat kernel given by

$$q_s(z,t) = c_n \int_{-\infty}^{\infty} e^{-i\lambda t} \left(\frac{\lambda}{\sinh \lambda s} \right)^n e^{-\frac{1}{4}(\lambda \coth \lambda s)|z|^2} d\lambda$$

with the positive constant c_n. Note that the heat kernel is a C^∞ function on $H^n \times (0, \infty)$ and its Fourier transform is (see p. 86 in [18])

$$\hat{q}_s(\lambda) = e^{-sH(\lambda)}.$$

More details about the sublaplacian and the heat kernel on Heisenberg group can be found in [16,18].

The Heisenberg Radon transform is defined by

$$Rf(z,t) = f * \delta_2(z,t) = \int_{\mathbb{C}^n} f((z,t)(w,0))dw,$$

where δ_2 is the Dirac delta function in second variable. This convolution operator is an auxiliary tool for studying other operators which received much attentions in the area of abstract harmonic analysis (see [5,6,8,13,19]).

3 Proof of the Main Result

In order to prove the main theorem, we first need some lemmas for preparation. Throughout the paper, we will use C to denote the positive constant, which is not necessarily same at each occurrence.

Now the following lemma is about the estimates for the differential operator $T = (\frac{i\partial}{\partial t})^n$ of the heat kernel:

Lemma 3.1 *The heat kernel $q_s(z,t)$ satisfies the estimate*

$$|Tq_s(z,t)| \le Cs^{-2n-1} e^{-A|(z,t)|^2/s}$$

with some positive constants C and A. Moreover, we have

$$\|Tq_s\|_1 \le Cs^{-n} \quad \text{and} \quad \|Tq_s\|_2 \le Cs^{-(3n+1)/2}.$$

Proof It is easy to compute that

$$\begin{aligned} Tq_s(z,t) &= c_n \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{\lambda^{2n}}{\sinh^n \lambda s} e^{-\frac{1}{4}(\lambda \coth \lambda s)|z|^2} d\lambda \\ &= c_n s^{-2n-1} \int_{-\infty}^{\infty} e^{-i\lambda t/s} \frac{\lambda^{2n}}{\sinh^n \lambda} e^{-\frac{1}{4}(\lambda \coth \lambda)|z|^2/s} d\lambda \\ &= c_n s^{-2n-1} K_s(z,t), \end{aligned}$$

where $K_s(z,t)=K_1(\frac{z}{\sqrt{s}},\frac{t}{s})$ and

$$K_1(z,t)=\int_{-\infty}^{\infty}e^{-i\lambda t}\frac{\lambda^{2n}}{\sinh^n\lambda}e^{-\frac{1}{4}(\lambda\coth\lambda)|z|^2}d\lambda.$$

On one hand, recall that when $|\lambda|<1,$ $\coth\lambda$ behaves like λ^{-1} and $\sinh\lambda$ behaves like λ; when $|\lambda|\geq 1,$ $\coth\lambda$ behaves like a constant and $\sinh\lambda$ behaves like e^{λ}. Hence we have

$$\begin{aligned}K_1(z,t)&\leq Ce^{-C'|z|^2}\int_{-\infty}^{\infty}\frac{\lambda^{2n}}{\sinh^n\lambda}d\lambda\\&\leq Ce^{-C'|z|^2}.\end{aligned}$$

On the other hand, for the ordinary Fourier transform one has $\mathcal{F}\big(\frac{1}{2\pi a}e^{-\frac{|z|^2}{4a}}\big)=e^{-a|\zeta|^2}$ $(a>0)$, then

$$\mathcal{F}K_1(\zeta,\lambda)=C\frac{\lambda^n}{\cosh^n\lambda}e^{-\frac{\tanh\lambda}{\lambda}|\zeta|^2}.$$

Note that this function can be extended to a holomorphic function of λ in the strip $\{\lambda-i\tau:|\tau|<\pi/2\}$. For $|\tau|\leq\pi/2-\delta$, $\mathcal{F}K_1(\zeta,\lambda-i\tau)$ is integrable in (ζ,λ) and rapidly decreasing in λ, uniformly in ζ and τ. Hence by a change of contour integration on the plane $\lambda-i\tau$ we get

$$\begin{aligned}K_1(z,t)&=C\int_{\mathbb{C}^n\times\mathbb{R}}\mathcal{F}K_1(\zeta,\lambda-i\tau)e^{-i\left(t(\lambda-i\tau)+\mathrm{Re}\langle z,\zeta\rangle\right)}d\zeta d\lambda\\&=Ce^{-t\tau}\int_{\mathbb{C}^n\times\mathbb{R}}\mathcal{F}K_1(\zeta,\lambda-i\tau)e^{-i\left(t\lambda+\mathrm{Re}\langle z,\zeta\rangle\right)}d\zeta d\lambda\\&\leq Ce^{-t\tau}\int_{\mathbb{C}^n\times\mathbb{R}}|\,\mathcal{F}K_1(\zeta,\lambda-i\tau)\,|\,d\zeta d\lambda\\&\leq Ce^{-t\tau}.\end{aligned}$$

From the discussion above and the inequality $(|z|^4+|t|^2)^{\frac{1}{2}}\leq|z|^2+|t|$, we obtain that, for some positive constant C and A,

$$|Tq_s(z,t)|\leq Cs^{-2n-1}e^{-A|(z,t)|^2/s}.$$

Now by (2.1) we have

$$\begin{aligned}\|Tq_s\|_1&\leq C\int_{H^n}s^{-2n-1}e^{-\frac{A}{s}|(z,t)|^2}dzdt\\&=C\int_0^{\infty}s^{-2n-1}e^{-\frac{Ar^2}{s}}r^{2n+1}dr\\&=Cs^{-n}.\end{aligned}$$

Similarly we get $\|Tq_s\|_2 \leq Cs^{-(3n+1)/2}$. As desired. □

Lemma 3.2 *Suppose* $f \in L^2(H^n)$. *Then*

$$\|TR(f * q_s)\|_2 = \|Rf * (Tq_s)\|_2.$$

Proof Since the Dirac delta function δ_2 and the heat kernel are both radial functions, then their convolution is commutative, i.e., $q_s * \delta_2 = \delta_2 * q_s$. Hence

$$\begin{aligned} TR(f * q_s) &= T\big((f * q_s) * \delta_2\big) \\ &= T\big(f * \delta_2 * q_s\big) \\ &= T\big(Rf * q_s\big) \\ &= Rf * (Tq_s). \end{aligned}$$

□

Lemma 3.3 *Suppose* $f \in L^2(H^n)$. *Then for* $0 \leq \gamma < n+1$, *one has*

$$\|Rf * Tq_s\|_2 \leq Cs^{-(n+\gamma/2)} \left(\int_{H^n} |(z,t)|^{2\gamma} |Rf(z,t)|^2 dzdt \right)^{1/2}.$$

Proof Let $B_r = B_r(0,0)$ and set $f_r = f\chi_{B_r}$, $f^r = f\chi_{H^n \setminus B_r}$. Note that

$$\|Rf * (Tq_s)\|_2 \leq \|(Rf)^r * (Tq_s)\|_2 + \|(Rf)_r * (Tq_s)\|_2 = A_1 + A_2.$$

By the generalized Young inequality together with the lemmas above we get

$$\begin{aligned} A_1 = \|(Rf)^r * (Tq_s)\|_2 &\leq \|(Rf)^r\|_2 \|Tq_s\|_1 \\ &\leq Cs^{-n} r^{-\gamma} \left(\int_{H^n \setminus B_r} |(z,t)|^{2\gamma} |Rf(z,t)|^2 dzdt \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \|(Rf)_r * (Tq_s)\|_2 \\ &\leq \|(Rf)_r\|_1 \|Tq_s\|_2 \\ &\leq Cs^{-\frac{3n+1}{2}} \left(\int_{H^n} |(z,t)|^{2\gamma} |Rf(z,t)|^2 dzdt \right)^{1/2} \left(\int_{B_r} |(z,t)|^{-2\gamma} dzdt \right)^{1/2}. \end{aligned}$$

Note that the integral $\int_{B_r} |(z,t)|^{-2\gamma} dzdt$ is controlled by $Cr^{-2\gamma+2n+2}$ as $0 \leq \gamma < n+1$. Therefore we have

$$\|Rf * (Tq_s)\|_2 \leq Cs^{-n} r^{-\gamma} (1 + s^{-\frac{n+1}{2}} r^{n+1}) \left(\int_{H^n} |(z,t)|^{2\gamma} |Rf(z,t)|^2 dzdt \right)^{1/2}.$$

Choosing $r = s^{1/2}$ then gives the desired estimate. $\square$

Lemma 3.4 *Suppose $f \in L^2(H^n)$ and $0 \le b \le 2$, one has*

$$\|f - f * q_s\|_2 \le C s^{b/2} \left(\int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2 dz dt \right)^{1/2}. \tag{3.1}$$

Proof By the Plancherel formula we have

$$\begin{aligned}
&\|f - f * q_s\|_2 \\
&= \left(\int_{-\infty}^{\infty} \sum_{\alpha} \|\hat{f}(\lambda)\big(1 - e^{-s(2|\alpha|+n)|\lambda|}\big) \Phi_\alpha^\lambda\|_2^2 d\mu(\lambda) \right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty} \sum_{\alpha} \|\hat{f}(\lambda) \frac{1 - e^{-s(2|\alpha|+n)|\lambda|}}{(s(2|\alpha|+n)|\lambda|)^{b/2}} (s(2|\alpha|+n)|\lambda|)^{b/2} \Phi_\alpha^\lambda\|_2^2 d\mu(\lambda) \right)^{1/2}.
\end{aligned}$$

Note that if $s \ge 1$, then for $b \ge 0$,

$$\frac{1 - e^{-s}}{s^{b/2}} \le 1;$$

if $0 < s < 1$, then $1 - e^{-s} \sim s (s \to 0^+)$ and thus for $0 \le b \le 2$,

$$\frac{1 - e^{-s}}{s^{b/2}} \le C.$$

Hence we get

$$\begin{aligned}
\|f - f * q_s\|_2 &\le C s^{b/2} \left(\int_{-\infty}^{\infty} \sum_{\alpha} \|\widehat{\mathscr{L}^{b/2} f}(\lambda) \Phi_\alpha^\lambda\|_2^2 d\mu(\lambda) \right)^{1/2} \\
&= C s^{b/2} \left(\int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2 dz dt \right)^{1/2},
\end{aligned}$$

where we have used the fact $\hat{f}(\lambda) H(\lambda)^{b/2} = \widehat{\mathscr{L}^{b/2} f}(\lambda)$ to get the last term. $\square$

Proof of Theorem 1.1 By Lemma 3.3 and Lemma 3.4 we have

$$\begin{aligned}
\|f\|_2 &\le \|f * q_s\|_2 + \|f - f * q_s\|_2 \\
&= \|TR(f * q_s)\|_2 + \|f - f * q_s\|_2 \\
&= \|Rf * (Tq_s)\|_2 + \|f - f * q_s\|_2 \\
&\le C s^{-(n+\gamma/2)} \left(\int_{H^n} |(z,t)|^{2\gamma} |Rf(z,t)|^2 dz dt \right)^{1/2} \\
&\quad + C s^{b/2} \left(\int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2 dz dt \right)^{1/2}.
\end{aligned}$$

Minimizing the right-hand side of the last inequality we then have for $0 \leq a < n+1,\ 0 \leq b \leq 2$,

$$\|f\|_2 \leq C \| |(\cdot)|^a Rf\|_2^{\frac{b}{a+b+2n}} \|\mathscr{L}^{b/2} f\|_2^{\frac{a+2n}{a+b+2n}}. \tag{3.2}$$

Now if $b > 2$ and $b' \leq 2$, we have for all $\epsilon > 0$,

$$\frac{|\lambda|^{b'}}{\epsilon^{b'}} \leq 1 + \frac{|\lambda|^b}{\epsilon^b},$$

which implies that

$$\|\mathscr{L}^{b'/2} f\|_2 \leq \epsilon^{b'} \|f\|_2 + \epsilon^{b'-b} \|\mathscr{L}^{b'/2} f\|_2.$$

Optimizing in ϵ then gives the Landaw–Kolmogorov inequality:

$$\|\mathscr{L}^{b'/2} f\|_2 \leq C \|f\|_2^{1-b'/b} \|\mathscr{L}^{b/2} f\|_2^{b'/b}.$$

Plugging this into (3.2) with b replaced by b' then gives the desired result. □

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